Weight enumerators, intersection enumerators, and Jacobi polynomials

K.Honma, T.Okabe, M.Oura

Abstract

The intersection enumerator and the Jacobi polynomial in an *arbitrary* genus for a binary code are introduced. Adding the weight enumerator into our discussion, we give the explicit relations among them and give some of their basic properties.

1 Introduction

The weight enumerator plays an important role in coding theory. Gleason [6] initiated the applications of the weight enumerator to the invariant theory of the finite groups and Broué-Enguehard [3] constructed modular forms from the weight enumerators. These results were generalized to higher genera [1, 8, 4, 16, 10]. In [15], Ozeki provided the new notion "Jacobi polynomial" of a code. He stated that this comes out of considerations on various invariants of codes [12, 13, 14] and on Jacobi theta-series [5]. In [11], the notion of the intersection enumerator is given for some computations of extremal codes. In the present paper, we discuss these polynomials in an *arbitrary* genus with future applications in mind and some results in [15] are generalized to the case in higher genera.

We shall recall the coding theory (*cf.* [9, 7]). In the present paper we restrict to the binary case. Let $\mathbf{F}_2 = \{0, 1\}$ be the field of two elements and

Keywords: code, weight enumerator, intersection enumerator, Jacobi polynomial
2010MSC: Primary 94B05, Secondary 05E15
Running Head: Weight enumerators, intersection enumerators, and Jacobi polynomials

$\mathbf{F}_2^n$ the n-dimensional vector space over $\mathbf{F}_2$ equipped with the usual inner product

$$u \cdot v = u_1v_1 + \cdots + u_nv_n,\ u = (u_1, \ldots, u_n),\ v = (v_1, \ldots, v_n) \in \mathbf{F}_2^n.$$

For $v_1 = (v_{11}, \ldots, v_{1n_1}) \in \mathbf{F}_2^{n_1},\ v_2 = (v_{21}, \ldots, v_{2n_2}) \in \mathbf{F}_2^{n_2}$, we put

$$v_1 \oplus v_2 = (v_{11}, \ldots, v_{1n_1}, v_{21}, \ldots, v_{2n_2}).$$

We introduce an operation $\circ$ on $\mathbf{F}_2^n$ which is given by

$$u \circ v = (u_1v_1, \ldots, u_nv_n).$$

This operation satisfies the associative law $(u_1 \circ u_2) \circ u_3 = u_1 \circ (u_2 \circ u_3)$. We denote by $e_{i_1 \ldots i_\ell}$ the element of $\mathbf{F}_2^g$, whose entry is 1 for the i_1-,..., i_ℓ-part and 0 otherwise. For example, $e_{12} = (1, 1, 0) \in \mathbf{F}_2^3$. Therefore every non-zero element of $\mathbf{F}_2^g$ can be expressed as $e_{i_1 \ldots i_\ell}$ for suitable $i_1, \ldots, i_\ell$. The zero vector in $\mathbf{F}_2^g$ is denoted by e_0. The weight $wt(u)$ is the number of non-zero coordinates of u. The intersection number $u * v$ in the sense of [15] is $wt(u \circ v)$ in this paper. We denote by $n_a(u_1, \ldots, u_g)$ the number of i such that $a = (u_{1i}, \ldots, u_{gi})$ for $a \in \mathbf{F}_2^g$.

A linear code of length n is a linear subspace of $\mathbf{F}_2^n$. We denote by $C^\perp$ the dual code of C:

$$C^\perp = \{u \in \mathbf{F}_2^n |\ u \cdot v = 0,\ \forall v \in C\}.$$

If $C = C^\perp$, then C is called self-dual. If $wt(u) \equiv 0 \pmod 4$, $\forall u \in C$, then C is called doubly even. It is known that a self-dual and doubly even code of length n exists if and only if $n \equiv 0 \pmod 8$.

For a code C_1 (resp. C_2) of length n_1 (resp. n_2), we denote by $C_1 \oplus C_2$ the direct sum of C_1 and C_2. In other words,

$$C_1 \oplus C_2 = \{u_1 \oplus u_2 \mid\ u_1 \in C_1,\ u_2 \in C_2\}.$$

The (homogeneous) weight enumerator of a code C of length n is

$$W_C(x, y) = \sum_{u \in C} x^{n - wt(u)} y^{wt(u)}.$$

This is the first weight enumerator which will be defined in the next section. The inhomogeneous weight enumerator of a code C of length n is[1] $W_C(X) = W_C(x \leftarrow 1, y \leftarrow X)$.

[1] The notation $x \leftarrow X$ means to substitute X for x.

2 Definitions of Polynomials of Codes

We start with giving the definitions of the three different kinds of polynomials of codes. It should be emphasized that the notion of the genus attaches to each polynomial.

Definition 1 *Let g be a positive integer and C be a code of length n.*

(1) *The g-th weight enumerator of C is*

$$W_C^{(g)}(\{x_a\}_{a\in\mathbf{F}_2^g}) = \sum_{u_1,\dots,u_g\in C}\prod_{a\in\mathbf{F}_2^g} x_a^{n_a(u_1,\dots,u_g)}.$$

(2) *The g-th Jacobi polynomial of C with the reference vector $v\in\mathbf{F}_2^n$ is*

$$\begin{aligned}&Jac^{(g)}(C,v|\ \{X_i\}_{1\le i\le g}, \{X_{k_1\dots k_\ell}\}_{\substack{2\le\ell\le g+1\\ 1\le k_1<\dots<k_\ell\le g+1}})\\ &= \sum_{u_1,\dots,u_g\in C}\left(\prod_{1\le i\le g} X_i^{wt(u_i)}\right)\left\{\prod_{2\le\ell\le g+1}\left(\prod_{\substack{1\le k_1<\dots<k_\ell\le g+1\\ u_{g+1}=v}} X_{k_1\dots k_\ell}^{wt(u_{k_1}\circ\dots\circ u_{k_\ell})}\right)\right\}.\end{aligned}$$

(3) *The g-th intersection enumerator of C is*

$$I_C^{(g)}(\{X_{k_1\dots k_\ell}\}_{\substack{1\le\ell\le g\\ 1\le k_1<\dots<k_\ell\le g}}) = \sum_{u_1,\dots,u_g\in C}\left\{\prod_{1\le\ell\le g}\left(\prod_{1\le k_1<\dots<k_\ell\le g} X_{k_1\dots k_\ell}^{wt(u_{k_1}\circ\dots\circ u_{k_\ell})}\right)\right\}.$$

When no confusion occurs, we omit the notation of variables in the polynomials and write as $W_C^{(g)}$, $Jac^{(g)}(C,v)$, and $I_C^{(g)}$. It is well known that the g-th weight enumerator is attracted by the number theory (*cf.* [3, 4, 16]).

Remark 2 (1) $W_C^{(1)}(x_0\leftarrow 1, x_1\leftarrow X) = Jac^{(1)}(C,0|\ X_1\leftarrow X, X_{12}) = I_C^{(1)}(X_1\leftarrow X)$.

(2) *The number of variables in each polynomial is given by*

$W_C^{(g)}$: 2^g,

$Jac^{(g)}(C,v)$: $g+\binom{g+1}{2}+\binom{g+1}{3}+\dots+\binom{g+1}{g+1} = 2^{g+1}-2$,

$I_C^{(g)}$: $\binom{g}{1}+\binom{g}{2}+\dots+\binom{g}{g} = 2^g-1$.

At this point, the difference between $Jac^{(g)}(C,v)$ and $I_C^{(g+1)}$ is that $I_C^{(g+1)}$ contains X_{g+1} in its definition, whereas $Jac^{(g)}(C,v)$ does not.

(3) $I_C^{(g)}(X_{i_1\dots i_\ell} \leftarrow 1$ *for* $\ell \geq 3)$ *is the intersection enumerator in genus* g *in the sense of* [11].

We shall give the explicit forms for $g = 1, 2$. For $g = 1$, we have

$$Jac^{(1)}(C, v|\ X_1 \leftarrow X, X_{12} \leftarrow Z) = \sum_{u \in C} X^{wt(u)} Z^{wt(u \circ v)}.$$

This is the Jacobi polynomial of binary codes dealt in [15]. For $g = 2$, we have

$$Jac^{(2)}(C, v) = \sum_{u_1, u_2 \in C} X_1^{wt(u_1)} X_2^{wt(u_2)} X_{12}^{wt(u_1 \circ u_2)} X_{13}^{wt(u_1 \circ u_3)} X_{23}^{wt(u_2 \circ u_3)} X_{123}^{wt(u_1 \circ u_2 \circ u_3)}$$

where $u_3 = v$ in the right-hand side.

3 Basic Results

The Jacobi polynomial $Jac^{(g)}(C, v)$ has the following expansion[2].

$$\begin{aligned} Jac^{(g)}(C, v) = \sum_{\substack{\{m_i\} \\ \{r_{k_1\dots k_\ell}\}}} & b(\{m_i\}_{1 \leq i \leq g}, \{r_{k_1\dots k_\ell}\}_{\substack{2 \leq \ell \leq g+1 \\ 1 \leq k_1 < \cdots < k_\ell \leq g+1}}) \left(\prod_{1 \leq i \leq g} X_i^{m_i} \right) \\ & \times \left\{ \prod_{2 \leq \ell \leq g+1} \left(\prod_{1 \leq k_1 < \cdots < k_\ell \leq g+1} X_{k_1\dots k_\ell}^{r_{k_1\dots k_\ell}} \right) \right\} \end{aligned}$$

where $b(\{m_i\}_{1 \leq i \leq g}, \{r_{k_1\dots k_\ell}\}_{\substack{2 \leq \ell \leq g+1 \\ 1 \leq k_1 < \cdots < k_\ell \leq g+1}})$ is the number of $(u_1, \dots, u_g) \in C^g$ satisfying

$$\begin{cases} wt(u_i) = m_i & (1 \leq i \leq g), \\ wt(u_{k_1} \circ \cdots \circ u_{k_\ell}) = r_{k_1\dots k_\ell} & (2 \leq \ell \leq g+1,\ 1 \leq k_1 < \cdots < k_\ell \leq g+1). \end{cases}$$

Here we consider as $u_{g+1} = v$. We have a trivial inequality $wt(u_{j_1} \circ \cdots \circ u_{j_{\ell+1}}) \leq wt(u_{i_1} \circ \cdots \circ u_{i_\ell})$ for $\{i_1, \dots, i_\ell\} \subset \{j_1, \dots, j_\ell, j_{\ell+1}\}$. From this observation, we have

[2]If we use the notation $r_i = wt(u_i)$ instead of m_i, some parts of the descriptions below might be simplified, however, we did not take that way because we need to emphasize the distinction between m_* and r_*. Also we respect the usage of the notation in Ozeki's original paper [15].

Proposition 3 *If there exists some pair* $\{i_1, \ldots, i_\ell\} \subset \{j_1, \ldots, j_\ell, j_{\ell+1}\}$ *for some* $\ell \geq 1$ *such that*

$$m_{i_1} < r_{j_1 j_2} \text{ for } \ell = 1$$

or

$$r_{i_1 \ldots i_\ell} < r_{j_1 \ldots j_\ell j_{\ell+1}} \text{ for } \ell \geq 2,$$

then we have $b(\{m_i\}, \{r_{i_1 \ldots i_\ell}\}) = 0$.

Here we are assuming the convention $u_{g+1} = v$ and $r_{g+1} = wt(u_{g+1})$.

Proposition 4 *Let* C *be a code of length* n *which contains all one vector* $\mathbf{1}$ *and* $Jac^{(g)}(C, v) = \sum b(\{m_i\}, \{r_{k_1 \ldots k_\ell}\}) \prod X_i^{m_i} \prod X_{k_1 \ldots k_\ell}^{r_{k_1 \ldots k_\ell}}$ *be the* g*-th Jacobi polynomial of* C *with the reference vector* v *of weight* k*. Fix* j $(1 \leq j \leq g)$*. Then it holds*

$$b(\{m_i\}, \{r_{k_1 \ldots k_\ell}\}) = b(\{m'_i\}, \{r'_{k_1 \ldots k_\ell}\})$$

where

$$m'_i = \begin{cases} m_i & \text{if } i \neq j, \\ n - m_j & \text{otherwise,} \end{cases}$$

and

$$r'_{k_1 \ldots k_\ell} = \begin{cases} r_{k_1 \ldots k_\ell} & \text{if } i \notin \{k_1, \ldots, k_\ell\}, \\ r_{k_1 \ldots \hat{i} \ldots k_\ell} - r_{k_1 \ldots i \ldots k_\ell} & \text{otherwise,} \end{cases}$$

and $\hat{i}$ *means to exclude* i.

Proof. The map $(u_1, \ldots, u_i, \ldots, u_g) \mapsto (u_1, \ldots, 1 - u_i, \ldots, u_g)$ is a bijection from $\{(u_1, \ldots, u_g) \in C^g \mid wt(u_{i_1} \circ \cdots \circ u_{i_\ell}) = r_{k_1 \ldots k_\ell}, \forall k_1, \ldots, k_\ell\}$ to $\{(u_1, \ldots, u_g) \in C^g \mid wt(u_{i_1} \circ \cdots \circ u_{i_\ell}) = r'_{k_1 \ldots k_\ell}, \forall k_1, \ldots, k_\ell\}$. This completes the proof of Proposition 4.

Making successive use of this proposition, we get, for $g = 2$,

$$\begin{aligned} & b(\{m_1, m_2\}, \{r_{12}, r_{13}, r_{23}, r_{123}\}) \\ & = b(\{n - m_1, m_2\}, \{m_2 - r_{12}, k - r_{13}, r_{23}, r_{23} - r_{123}\}) \\ & = b(\{m_1, n - m_2\}, \{m_1 - r_{12}, r_{13}, k - r_{23}, r_{13} - r_{123}\}) \\ & = b(\{n - m_1, n - m_2\}, \{n - m_1 - m_2 + r_{12}, k - r_{13}, k - r_{23}, k - r_{13} - r_{23} + r_{123}\}). \end{aligned}$$

Next, we shall consider the reduction of the intersection enumerator to the inhomogeneous weight enumerator.

Proposition 5 (1) $I_C^{(g+1)}(X_{k_1 \ldots k_\ell} \leftarrow 1 \textit{ for } \ell \geq 2) = \prod_{i=1}^{g+1} W_C(X_i)$.

(2) $I_C^{(g+1)}(\textit{variables} \leftarrow 1 \textit{ except } X_i) = |C|^g W_C(X_i)$.

Proof. For (1), we have

$$\begin{aligned} I_C^{(g+1)}(X_{k_1 \ldots k_\ell} \leftarrow 1 \text{ for } \ell \geq 2) &= \sum_{u_1,\ldots,u_{g+1} \in C^{g+1}} \left(\prod_{1 \leq i \leq g+1} X_i^{wt(u_i)} \right) \left\{ \prod_{2 \leq \ell \leq g+1} \left(\prod_{1 \leq k_1 < \cdots < k_\ell \leq g+1} 1 \right) \right\} \\ &= \sum_{u_1,\ldots,u_{g+1} \in C^{g+1}} \left(\prod_{1 \leq i \leq g+1} X_i^{wt(u_i)} \right) \\ &= \prod_{1 \leq i \leq g+1} \left(\sum_{u_i \in C} X_i^{wt(u_i)} \right) \\ &= \prod_{1 \leq i \leq g+1} W_C(X_i). \end{aligned}$$

The assertion (2) follows from (1) by specializing variables. This completes the proof of Proposition 5.

We would like to provide the relationship between the weight enumerator and the intersection enumerator. In order to do this, we require the following

Lemma 6 *Let $u_1, \ldots, u_g$ be elements of $\mathbf{F}_2^n$. Then the following hold for $1 \leq \ell \leq g$.*

(1) $wt(u_{i_1} \circ \cdots \circ u_{i_\ell}) = \sum_{\substack{\{i_1,\ldots,i_\ell\} \subset \{j_1,\ldots,j_k\} \\ 1 \leq j_1 < \cdots < j_k \leq g}} n_{e_{j_1 \cdots j_k}}(u_1, \ldots, u_g)$.

(2) $n_{e_{i_1 \cdots i_\ell}}(u_1, \ldots, u_g) = \sum_{\substack{\{i_1,\ldots,i_\ell\} \subset \{j_1,\ldots,j_k\} \\ 1 \leq j_1 < \cdots < j_k \leq g}} (-1)^{k-\ell} wt(u_{j_1} \circ \cdots \circ u_{j_k})$.

Proof. (1) The right-hand side counts the coordinate positions m such that

$$u_{i_1,m} = u_{i_2,m} = \cdots = u_{i_\ell,m} = 1.$$

This is $wt(u_{i_1} \circ \cdots \circ u_{i_\ell})$.

(2) The right-hand side is equal to

$$\sum_{\substack{\{i_1,\ldots,i_\ell\}\subset\{j_1,\ldots,j_k\}\\ 1\le j_1<\cdots<j_k\le g}} (-1)^{k-\ell} \left\{ \sum_{\substack{\{j_1,\ldots,j_k\}\subset\{m_1,\ldots,m_s\}\\ 1\le m_1<\cdots<m_s\le g}} n_{e_{m_1\ldots m_s}}(u_1,\ldots,u_g) \right\}.$$

We examine each term in this sum. It is easy to see that $n_{e_{i_1\ldots i_\ell}}(u_1,\ldots,u_g)$ appears only once. We calculate the coefficient of $n_{e_{m_1\ldots m_s}}(u_1,\ldots,u_g)$ where $\{i_1,\ldots,i_\ell\} \subsetneqq \{m_1,\ldots,m_s\}$. The coefficient is

$$\sum_{k=\ell}^{s} |\{\{j_1,\ldots,j_k\}:\ \{i_1,\ldots,i_\ell\}\subset\{j_1,\ldots,j_k\}\subset\{m_1,\ldots,m_s\}\}|$$

and this is

$$\begin{aligned}\sum_{k=\ell}^{s}(-1)^{k-\ell}\binom{s}{k-\ell} &= \sum_{t=0}^{s-\ell}(-1)^t\binom{s-\ell}{t}\\ &= (1+(-1))^{s-\ell}\\ &= 0.\end{aligned}$$

Thus, the sum under discussion turns out to consist of essentially only one term, that is, $n_{e_{i_1\ldots i_\ell}}(u_1,\ldots,u_g)$. This completes the proof of Lemma 6.

Now, we discuss the relations between the weight enumerator and the intersection enumerator.

Theorem 7 *Let C be a code of length n. Then the following hold.*

(1)

$$W_C^{(g)}\left(x_{e_0}\leftarrow 1, x_{e_{j_1\cdots j_k}} \leftarrow \prod_{\substack{\{i_1,\ldots,i_\ell\}\subset\{j_1,\ldots,j_k\}\\ 1\le j_1<\cdots<j_k\le g}} X_{i_1\ldots i_\ell} \text{ for } 1\le k\le g \right)$$
$$= I_C^{(g)}(\{X_{i_1 i_2\ldots i_\ell}\}_{\substack{1\le\ell\le g\\ 1\le i_1<i_2<\cdots<i_\ell\le g}}).$$

(2) $$x_{e_0}^n I_C^{(g)}\left(X_{j_1\cdots j_k} \leftarrow x_{e_0}^{(-1)^k} \prod_{\substack{\{i_1,\ldots,i_\ell\}\subset\{j_1,\ldots,j_k\}\\ 1\le i_1<\cdots<i_\ell\le g}} x_{e_{i_1}\cdots e_{i_\ell}}^{(-1)^{k-\ell}} \ for\ 1\le k\le g\right) = W_C^{(g)}(\{x_a\}_{a\in\mathbf{F}_2^g}).$$

Proof. (1) We have that

$$\begin{aligned}
&W_C^{(g)}\left(x_{e_0}\leftarrow 1,\ x_{e_{j_1\cdots j_k}} \leftarrow \prod_{\substack{\{i_1,\ldots,i_\ell\}\subset\{j_1,\ldots,j_k\}\\ 1\le i_1<\cdots<i_\ell\le g}} X_{i_1\ldots i_\ell}\right)\\
&\quad= \sum_{u_1,\ldots,u_g\in C}\prod_{1\le j_1<\cdots<j_k\le g}\left(\prod_{\substack{\{i_1,\ldots,i_\ell\}\subset\{j_1,\ldots,j_k\}\\ 1\le i_1<\cdots<i_\ell\le g}} X_{i_1\ldots i_\ell}\right)^{e_{j_1\cdots j_k}(u_1,\ldots,u_g)}\\
&\quad= \sum_{u_1,\ldots,u_g\in C}\prod_{1\le j_1<\cdots<j_k\le g} X_{i_1\ldots i_\ell}^{\sum_{\substack{\{i_1,\ldots,i_\ell\}\subset\{j_1,\ldots,j_k\}\\ 1\le i_1<\cdots<i_\ell\le g}} n_{e_{j_1\cdots j_k}}}\\
&\quad= \sum_{u_1,\ldots,u_g\in C}\prod_{1\le j_1<\cdots<j_k\le g} X_{i_1\ldots i_\ell}^{wt(u_{i_1}\circ\cdots\circ u_{i_\ell})}\\
&\quad= I_C^{(g)}.
\end{aligned}$$

(2) We first observe a trivial relation

$$n_{e_0}(u_1,\ldots,u_g) = n - \sum_{1\le i_1<\cdots<i_\ell\le g} n_{e_{i_1\cdots i_\ell}}(u_1,\ldots,u_g).$$

Then we have that

$$
\begin{aligned}
W_C^{(g)} &= \sum_{u_1,\ldots,u_g \in C} x_{e_0}^{n_{e_0}(u_1,\ldots,u_g)} \prod_{1 \le i_1 < \cdots < i_\ell \le g} x_{e_{i_1 \ldots i_\ell}}^{n_{e_1 \ldots e_\ell}(u_1,\ldots,u_g)} \\
&= \sum_{u_1,\ldots,u_g \in C} x_{e_0}^{n - \sum_{1 \le i_1 < \cdots < i_\ell \le g} n_{e_{i_1 \ldots i_\ell}}(u_1,\ldots,u_g)} \prod_{1 \le i_1 < \cdots < i_\ell \le g} x_{e_{i_1 \ldots i_\ell}}^{n_{e_1 \ldots e_\ell}(u_1,\ldots,u_g)} \\
&= x_{e_0}^n \sum_{u_1,\ldots,u_g \in C} \prod_{1 \le i_1 < \cdots < i_\ell \le g} \left(\frac{x_{e_{i_1 \ldots i_\ell}}}{x_{e_0}} \right)^{n_{e_{i_1 \ldots i_\ell}}(u_1,\ldots,u_g)} \\
&= x_{e_0}^n \sum_{u_1,\ldots,u_g \in C} \prod_{1 \le i_1 < \cdots < i_\ell \le g} \left(\frac{x_{e_{i_1 \ldots i_\ell}}}{x_{e_0}} \right)^{\sum_{\substack{\{i_1,\ldots,i_\ell\} \subset \{j_1,\ldots,j_k\} \\ 1 \le j_1 < \cdots < j_k \le g}} (-1)^{k-\ell} wt(u_{j_1} \circ \cdots \circ u_{j_k})} \\
&= x_{e_0}^n \sum_{u_1,\ldots,u_g \in C} \prod_{1 \le j_1 < \cdots < j_k \le g} \left\{ \prod_{\substack{\{i_1,\ldots,i_\ell\} \subset \{j_1,\ldots,j_k\} \\ 1 \le i_1 < \cdots < i_\ell \le g}} \left(\frac{x_{e_{i_1 \ldots i_\ell}}}{x_{e_0}} \right)^{(-1)^{k-\ell}} \right\}^{wt(u_{j_1} \circ \cdots \circ u_{j_k})} .
\end{aligned}
$$

Making use of

$$
\begin{aligned}
\prod_{\substack{\{i_1,\ldots,i_\ell\} \subset \{j_1,\ldots,j_k\} \\ 1 \le i_1 < \cdots < i_\ell \le g}} \left(\frac{1}{x_{e_0}} \right)^{(-1)^{k-\ell}} &= \left(\frac{1}{x_{e_0}} \right)^{(-1)^{k-1}\binom{k}{1} + (-1)^{k-2}\binom{k}{2} + \cdots + (-1)^{k-k}\binom{k}{k}} \\
&= \left(\frac{1}{x_{e_0}} \right)^{-(-1)^k} \\
&= x_{e_0}^{(-1)^k},
\end{aligned}
$$

we continue

$$
\begin{aligned}
& x_{e_0}^n \sum_{u_1,\ldots,u_g \in C} \prod_{1 \le j_1 < \cdots < j_k \le g} \left(x_{e_0}^{(-1)^k} \prod_{\substack{\{i_1,\ldots,i_\ell\} \subset \{j_1,\ldots,j_k\} \\ 1 \le i_1 < \cdots < i_\ell \le g}} x_{e_{i_1 \ldots i_\ell}}^{(-1)^{k-\ell}} \right)^{wt(u_{j_1} \circ \cdots \circ u_{j_k})} \\
& \quad = x_{e_0}^n I_C^{(g)} \left(X_{j_1 \ldots j_k} \leftarrow x_{e_0}^{(-1)^k} \prod_{\substack{\{i_1,\ldots,i_\ell\} \subset \{j_1,\ldots,j_k\} \\ 1 \le i_1 < \cdots < i_\ell \le g}} x_{e_{i_1} \ldots e_{i_\ell}}^{(-1)^{k-\ell}} \text{ for } 1 \le k \le g \right).
\end{aligned}
$$

This completes the proof of Theorem 7.

We give some examples.

$$W_C^{(2)}(x_{e_0} \leftarrow 1, x_{e_2} \leftarrow X_2, x_{e_1} \leftarrow X_1, x_{e_{12}} \leftarrow X_1X_2X_{12}) = I_C^{(2)}(X_1, X_2, X_{12}).$$

$$\begin{aligned}W_C^{(3)}(&x_{e_0} \leftarrow 1, x_{e_1} \leftarrow X_1, x_{e_2} \leftarrow X_2, x_{e_3} \leftarrow X_3,\\ &x_{e_{12}} \leftarrow X_1X_2X_{12}, x_{e_{13}} \leftarrow X_1X_3X_{13}, x_{e_{23}} \leftarrow X_2X_3X_{23}, x_{e_{123}} \leftarrow X_1X_2X_3X_{12}X_{13}X_{23}X_{123})\\ &= I_C^{(3)}(X_1, X_2, X_3, X_{12}, X_{13}, X_{23}, X_{123}).\end{aligned}$$

$$x_{e_0}^n I_C^{(2)}(X_1 \leftarrow \frac{x_{e_1}}{x_{e_0}}, X_2 \leftarrow \frac{x_{e_2}}{x_{e_0}}, X_{12} \leftarrow \frac{x_{e_0}x_{e_{12}}}{x_{e_1}x_{e_2}}) = W_C^{(2)}(x_{e_0}, x_{e_1}, x_{e_2}, x_{e_{12}}).$$

$$\begin{aligned}x_{e_0}^n I_C^{(3)}(&X_1 \leftarrow \frac{x_{e_1}}{x_{e_0}}, X_2 \leftarrow \frac{x_{e_2}}{x_{e_0}}, X_3 \leftarrow \frac{x_{e_3}}{x_{e_0}}, X_{12} \leftarrow \frac{x_{e_0}x_{e_{12}}}{x_{e_1}x_{e_2}}, X_{13} \leftarrow \frac{x_{e_0}x_{e_{13}}}{x_{e_1}x_{e_3}}, X_{23} \leftarrow \frac{x_{e_0}x_{e_{23}}}{x_{e_2}x_{e_3}},\\ &X_{123} \leftarrow \frac{x_{e_1}x_{e_2}x_{e_3}x_{e_{123}}}{x_{e_0}x_{e_{12}}x_{e_{13}}x_{e_{23}}})\\ &= W_C^{(3)}(x_{e_0}, x_{e_1}, x_{e_2}, x_{e_{12}}, x_{e_3}, x_{e_{13}}, x_{e_{23}}, x_{e_{123}}).\end{aligned}$$

It is known that the Jacobi forms appear in the Fourier-Jacobi expansion of Siegel modular forms (*cf.* [5, 17, 18]). This is one of the motivations for Ozeki for conducting studies on Jacobi polynomials. To get similar results for the g-th Jacobi polynomial, we consider the $(g+1)$-th intersection enumerator. For a code of length n, it holds

$$\begin{aligned}I_C^{(g+1)} &= \sum_{u_1,\dots,u_{g+1}\in C} X_1^{wt(u_1)}\dots X_g^{wt(u_g)} X_{g+1}^{wt(u_{g+1})} \prod_{2\le \ell\le g+1}\left(\prod_{1\le k_1<\dots<k_\ell\le g+1} X_{k_1\dots k_\ell}^{wt(u_{k_1}\circ\cdots\circ u_{k_\ell})}\right)\\ &= \sum_{u_{g+1}\in C}\left\{\sum_{u_1,\dots,u_g\in C}\prod_{2\le \ell\le g+1}\left(\prod_{1\le k_1<\dots<k_\ell\le g+1} X_{k_1\dots k_\ell}^{wt(u_{k_1}\circ\cdots\circ u_{k_\ell})}\right)\right\} X^{wt(u_{g+1})}\\ &= \sum_{u_{g+1}\in C} Jac^{(g)}(C, u_{g+1}) X_{g+1}^{wt(u_{g+1})}\\ &= \sum_{r=0}^{n}\left(\sum_{\substack{v\in C\\ wt(v)=r}} Jac^{(g)}(C, v)\right) X_{g+1}^r.\end{aligned}$$

We have thus obtained the following

Theorem 8 *Let C be a code of length n. Assuming that $X_{g+1} = Y$, we have*

$$I_C^{(g+1)} = \sum_{r=0}^{n} \left(\sum_{\substack{v \in C \\ wt(v)=r}} Jac^{(g)}(C, v) \right) Y^r.$$

In this theorem, the g-th Jacobi polynomial is obtained from the $(g+1)$-th weight enumerator. If we ask any relation between those of the same genus, we get the following

Proposition 9 *The g-th Jacobi polynomial of a code C with the zero reference vector is the g-th intersection enumerator, that is,*

$$Jac^{(g)}(C, 0) = I_C^{(g)}.$$

Proof. Since we have $wt(u_{k_1} \circ \cdots \circ u_{k_\ell}) = 0$ for $k_\ell = g+1$ from our assumption, it follows that

$$\prod_{2 \le \ell \le g+1} \left(\prod_{\substack{1 \le k_1 < \cdots < k_\ell \le g+1 \\ u_{g+1} = v}} X_{k_1 \ldots k_\ell}^{wt(u_{k_1} \circ \cdots \circ u_{k_\ell})} \right) = \prod_{2 \le \ell \le g} \left(\prod_{1 \le k_1 < \cdots < k_\ell \le g} X_{k_1 \ldots k_\ell}^{wt(u_{k_1} \circ \cdots \circ u_{k_\ell})} \right).$$

Therefore we have

$$\begin{aligned} Jac^{(g)}(C, 0) &= \sum_{u_1, \ldots, u_g \in C} \left(\prod_{1 \le i \le g} X_i^{wt(u_i)} \right) \left\{ \prod_{2 \le \ell \le g+1} \left(\prod_{\substack{1 \le k_1 < \cdots < k_\ell \le g+1 \\ u_{g+1} = v}} X_{k_1 \ldots k_\ell}^{wt(u_{k_1} \circ \cdots \circ u_{k_\ell})} \right) \right\} \\ &= \sum_{u_1, \ldots, u_g \in C} \left(\prod_{1 \le i \le g} X_i^{wt(u_i)} \right) \left\{ \prod_{2 \le \ell \le g} \left(\prod_{1 \le k_1 < \cdots < k_\ell \le g} X_{k_1 \ldots k_\ell}^{wt(u_{k_1} \circ \cdots \circ u_{k_\ell})} \right) \right\} \\ &= I_C^{(g)}. \end{aligned}$$

This completes the proof of Proposition 9.

Proposition 10 *For $i = 1, 2$, let C_i be a code of length n_i and v_i an element of $\mathbf{F}_2^{n_i}$. Then, we have*

$$Jac^{(g)}(C_1, v_1) Jac^{(g)}(C_2, v_2) = Jac^{(g)}(C_1 \oplus C_2, v_1 \oplus v_2).$$

Proof. The proof is straightforward.

We conclude this section with some explicit calculations. Let H be a code of length 8 defined by the generator matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

This is self-dual and doubly even. For simplicity, we set

$$a = X_1, b = X_2, c = X_{12}, d = X_{13}, e = X_{23}, f = X_{123}.$$

We take the following vectors as the reference vectors:

$$v_1 = (0,0,0,0,0,0,0,0),\ v_2 = (1,1,1,1,0,0,0,0),\ v_3 = (1,0,0,0,0,0,0,0).$$

The terms are grouped following the calculation made after Proposition 4.

$$Jac^{(2)}(H, v_1) = I_H^{(2)} =$$
$$(a^8b^8c^8+a^8+b^8+1)+14(a^8b^4c^4+b^4)+14(a^4b^8c^4+a^4)+14(a^4b^4c^4+a^4b^4)+168a^4b^4c^2.$$

$$\begin{aligned} Jac^{(2)}(H, v_2) &= (a^8b^8c^8d^4e^4f^4+b^8e^4+a^8d^4+1)+(a^8b^4c^4d^4e^4f^4+b^4e^4+a^8b^4c^4d^4+b^4) \\ &+ 12(a^8b^4c^4d^4e^2f^2 + b^4e^2) + (a^4b^8c^4d^4e^4f^4 + a^4b^8c^4e^4 + a^4d^4 + a^4) \\ &+ 12(a^4b^8c^4d^2e^4f^2 + a^4d^2) + (a^4b^4c^4d^4e^4f^4 + a^4b^4e^4 + a^4b^4d^4 + a^4b^4c^4) \\ &+ 12(a^4b^4c^4d^2e^2f^2 + a^4b^4d^2e^2) + 12(a^4b^4c^2d^4e^2f^2 + a^4b^4c^2e^2) \\ &+12(a^4b^4c^2d^2e^4f^2+a^4b^4c^2d^2)+12(a^4b^4c^2d^2e^2f^2+a^4b^4c^2d^2e^2)+96a^4b^4c^2d^2e^2f. \end{aligned}$$

$$\begin{aligned} Jac^{(2)}(H, v_3) &= (a^8b^8c^8def+b^8e+a^8d+1)+7(a^8b^4c^4def+b^4e+a^8b^4c^4d+b^4) \\ &+ 7(a^4b^8c^4def + a^4b^8c^4e + a^4d + a^4) + 7(a^4b^4c^4def + a^4b^4e + a^4b^4d + a^4b^4c^4) \\ &+ 42(a^4b^4c^2def + a^4b^4c^2e + a^4b^4c^2d + a^4b^4c^2). \end{aligned}$$

4 MacWilliams Idenity for Inhomogeneous Jacobi Polynomial

In this section, we give the MacWilliams identity for the inhomogeneous Jacobi polynomial of a code.

Theorem 11 *For a code C of length n, we have that*

$$Jac^{(g)}(C^{\perp}, v|\ \{X_i\}_{1\le i\le g}, \{X_{k_1\dots k_\ell}\}_{\substack{2\le \ell\le g+1\\ 1\le k_1<\dots<k_\ell\le g+1}})$$
$$= \frac{1}{|C|^g} Z_0^n \left(\frac{Z_{g+1}}{Z_0}\right)^{wt(v)} Jac^{(g)}(C, v|\ \left\{\frac{Z_i}{Z_0}\right\}_{1\le i\le g}, \{Z_{k_1\dots k_\ell}\}_{\substack{2\le \ell\le g+1\\ 1\le k_1<\dots<k_\ell\le g+1}})$$

where

$$Z_0 = \sum_{\substack{b_1,\dots,b_g\in \mathbf{F}_2\\ b_{g+1}=0}} \left(X_1^{b_1}\dots X_g^{b_g} \prod_{\substack{2\le t\le g+1\\ 1\le m_1<\dots<m_t\le g+1}} X_{m_1\dots m_t}^{b_{m_1}\dots b_{m_t}} \right),$$

$$Z_i = \sum_{\substack{b_1,\dots,b_g\in \mathbf{F}_2\\ b_{g+1}=0}} \left((-1)^{b_i} X_1^{b_1}\dots X_g^{b_g} \prod_{\substack{2\le t\le g+1\\ 1\le m_1<\dots<m_t\le g+1}} X_{m_1\dots m_t}^{b_{m_1}\dots b_{m_t}} \right) \quad (1\le i\le g),$$

$$Z_{g+1} = \sum_{\substack{b_1,\dots,b_g\in \mathbf{F}_2\\ b_{g+1}=1}} \left(X_1^{b_1}\dots X_g^{b_g} \prod_{\substack{2\le t\le g+1\\ 1\le m_1<\dots<m_t\le g+1}} X_{m_1\dots m_t}^{b_{m_1}\dots b_{m_t}} \right)$$

and for $\ell \ge 2$

$$Z_{k_1\dots k_\ell} = \prod_{\substack{0\le j\le \ell\\ 1\le i_1<\dots<i_j\le \ell\\ \{k_{i_1},\dots,k_{i_j}\}\subset\{k_1,\dots,k_\ell\}}} \left(\sum_{\substack{b_1,\dots,b_g\in \mathbf{F}_2\\ b_{g+1}=\delta_{k_{i_j},g+1}}} (-1)^{b_{k_{i_1}}+\dots+b_{k_{i_{j-1}}}+(1-b_{g+1})b_{k_{i_j}}} \cdot X_1^{b_1}\dots X_g^{b_g} \prod_{\substack{2\le t\le g+1\\ 1\le m_1<\dots<m_t\le g+1}} X_{m_1\dots m_t}^{b_{m_1}\dots b_{m_t}} \right)^{(-1)^{\ell-j}}.$$

Before proceeding to the proof, we give a remark concerning to the definition of $Z_{k_1\dots k_\ell}$. If $j=0$ which corresponds to $\emptyset \subset \{k_1,\dots,k_\ell\}$, the product factor is understood as Z_0. The integer $b_{g+1}=\delta_{k_{i_j},g+1}$ is a constant and we need to distinguish the cases $k_{i_j} = g+1$ and $k_{i_j} < g+1$. We note

$$(b_{g+1}-1)k_{i_j} = \begin{cases} k_{i_j} & (k_{i_j} < g+1), \\ 0 & (k_{i_j} = g+1). \end{cases}$$

Proof of Theorem 11. We follow the usual method of proving the MacWilliams identity with the function

$$\begin{aligned} \delta_{C^\perp}(w) &= \begin{cases} 1 & w \in C^\perp \\ 0 & w \notin C^\perp \end{cases} \\ &= \frac{1}{|C|}\sum_{u\in C}(-1)^{u\cdot w} \end{aligned}$$

for $w \in \mathbf{F}_2^n$. We have that

$$\begin{aligned} Jac^{(g)}(C^\perp, v) &= \sum_{v_1,\dots,v_g \in C} \left(\prod X_i^{wt(v_i)}\right)\left(\prod X_{k_1\dots k_\ell}^{wt(v_{k_1}\circ\cdots\circ v_{k_\ell})}\right) \\ &= \sum_{w_1,\dots,w_g \in \mathbf{F}_2^n} \left(\prod \delta_{C^\perp}(w_i) X_i^{wt(w_i)}\right)\left(\prod X_{k_1\dots k_\ell}^{wt(w_{k_1}\circ\cdots\circ w_{k_\ell})}\right) \\ &= \sum_{w_1,\dots,w_g \in \mathbf{F}_2^n} \left(\prod \frac{1}{|C|}\sum_{u_i\in C}(-1)^{u_i\cdot w_i} X_i^{wt(w_i)}\right)\left(\prod X_{k_1\dots k_\ell}^{wt(w_{k_1}\circ\cdots\circ w_{k_\ell})}\right) \\ &= \frac{1}{|C|^g} \sum_{\substack{u_1,\dots,u_g\in C \\ w_1,\dots,w_g \in \mathbf{F}_2^n \\ w_{g+1}=v}} (-1)^{(u_{11}w_{11}+\cdots+u_{1n}w_{1n})+\cdots+(u_{g1}w_{g1}+\cdots+u_{gn}w_{gn})} \\ &\cdot X_1^{w_{11}} \dots X_1^{w_{1n}} \dots X_g^{w_{g1}} \dots X_g^{w_{gn}} \\ &\cdot X_{12}^{w_{11}w_{21}} \dots X_{12}^{w_{1n}w_{2n}} \dots \\ &\cdot X_{k_1\dots k_\ell}^{w_{k_1 1}\dots w_{k_\ell 1}} \dots X_{k_1\dots k_\ell}^{w_{k_1 n}\dots w_{k_\ell n}} \\ &\cdot X_{12\dots g,g+1}^{w_{11}w_{21}\dots w_{g1}w_{g+1,1}} \dots X_{12\dots g,g+1}^{w_{1n}w_{2n}\dots w_{gn}w_{g+1,n}} \end{aligned}$$

$$= \frac{1}{|C|^g} \sum_{u_1,\dots,u_g \in C}$$

$$\prod_{1\le i\le g} \left\{ \sum_{w_{1i},\dots,w_{gi}\in \mathbf{F}_2} (-1)^{u_{1i}w_{1i}+\cdots+u_{gi}w_{gi}} \left(X_1^{w_{1i}} \dots X_g^{w_{gi}} \prod_{\substack{2\le \ell\le g+1\\ 1\le k_1<\cdots<k_\ell\le g+1}} X_{k_1\dots k_\ell}^{w_{k_1 i}\dots w_{k_\ell i}} \right) \right\}$$

$$= \frac{1}{|C|^g} \sum_{u_1,\dots,u_g \in C}$$

$$\prod_{a\in \mathbf{F}_2^{g+1}} \left\{ \sum_{\substack{b_1,\dots,b_g\in \mathbf{F}_2\\ b_{g+1}=a_{g+1}}} (-1)^{a_1b_1+\cdots+a_gb_g} \left(X_1^{b_1} \dots X_g^{b_g} \prod_{\substack{2\le \ell\le g+1\\ 1\le k_1<\cdots<k_\ell\le g+1}} X_{k_1\dots k_\ell}^{b_{k_1}\dots b_{k_\ell}} \right) \right\}^{n_a(u_1,\dots,u_g,v)}$$

$$= \frac{1}{|C|^g} \sum_{u_1,\dots,u_g \in C} \prod_{a\in \mathbf{F}_2^{g+1}} x_a^{n_a(u_1,\dots,u_g,v)}$$

where

$$x_a = \sum_{\substack{b_1,\dots,b_g\in \mathbf{F}_2\\ b_{g+1}=a_{g+1}}} (-1)^{a_1b_1+\cdots+a_gb_g} X_1^{b_1} \dots X_g^{b_g} \prod_{\substack{2\le \ell\le g+1\\ 1\le k_1<\cdots<k_\ell\le g+1}} X_{k_1\dots k_\ell}^{b_{k_1}\dots b_{k_\ell}}.$$

We observe the identity

$$\prod_{a\in \mathbf{F}_2^{g+1}} x_a^{n_a(u_1,\dots,u_g,v)} = x_0^n \prod_{\substack{1\le \ell\le g+1\\ 1\le k_1<\cdots<k_\ell\le g+1}} \left\{ \prod_{\substack{0\le j\le \ell\\ 1\le i_1<\cdots<i_j\le \ell\\ \{k_{i_1},\dots,k_{i_j}\}\subset\{k_1,\dots,k_\ell\}}} x_{k_{i_1}\dots k_{i_j}}^{(-1)^{\ell-j}} \right\}^{wt(u_{k_1}\circ\cdots\circ u_{k_\ell})}$$

by Lemma 6 (2). Therefore we have

$$
\begin{aligned}
Jac(C^{\perp}, v) &= \frac{1}{|C|^g} \sum_{u_1,\dots,u_g \in C} \prod_{a \in \mathbf{F}_2^{g+1}} x_a^{n_a(u_1,\dots,u_g,v)} \\
&= \frac{1}{|C|^g} \sum_{u_1,\dots,u_g \in C} x_0^n \prod_{\substack{1 \le \ell \le g+1 \\ 1 \le k_1 < \cdots < k_\ell \le g+1}} \left\{ \prod_{\substack{0 \le j \le \ell \\ 1 \le i_1 < \cdots < i_j \le \ell \\ \{k_{i_1},\dots,k_{i_j}\} \subset \{k_1,\dots,k_\ell\}}} x_{k_{i_1}\dots k_{i_j}}^{(-1)^{\ell-j}} \right\}^{wt(u_{k_1} \circ \cdots \circ u_{k_\ell})} \\
&= \frac{1}{|C|^g} x_0^n \left(\frac{x_{g+1}}{x_0} \right)^{wt(v)} \\
&\cdot \sum_{u_1,\dots,u_g \in C} \prod_{1 \le i \le g} \left(\frac{x_i}{x_0} \right)^{wt(u_i)} \prod_{\substack{2 \le \ell \le g+1 \\ 1 \le k_1 < \cdots < k_\ell \le g+1}} \left\{ \prod_{\substack{0 \le j \le \ell \\ 1 \le i_1 < \cdots < i_j \le \ell \\ \{k_{i_1},\dots,k_{i_j}\} \subset \{k_1,\dots,k_\ell\}}} x_{k_{i_1}\dots k_{i_j}}^{(-1)^{\ell-j}} \right\}^{wt(u_{k_1} \circ \cdots \circ u_{k_\ell})} .
\end{aligned}
$$

This is the expected formula stated in Theorem 11 if we put

$$Z_i = x_i \ (0 \le i \le g+1),$$

and for $\ell \ge 2$

$$Z_{k_1 \dots k_\ell} = \prod_{\substack{2 \le \ell \le g+1 \\ 1 \le k_1 < \cdots < k_\ell \le g+1}} \left\{ \prod_{\substack{0 \le j \le \ell \\ 1 \le i_1 < \cdots < i_j \le \ell \\ \{k_{i_1},\dots,k_{i_j}\} \subset \{k_1,\dots,k_\ell\}}} x_{k_{i_1}\dots k_{i_j}}^{(-1)^{\ell-j}} \right\}.$$

This completes the proof of Theorem 11.

We give the explicit forms for small genera.
For $g = 1$, we have that

$$
\begin{aligned}
x_0 &= 1 + X_1, \\
x_1 &= 1 - X_1, \\
x_2 &= 1 + X_1 X_{12}, \\
x_{12} &= 1 - X_1 X_{12}
\end{aligned}
$$

and

$$Z_i = x_i,\ Z_{12} = \frac{x_0 x_{12}}{x_1 x_2}.$$

For $g = 2$, we have that

$$\begin{aligned}
x_0 &= 1 + X_1 + X_2 + X_1X_2X_{12},\\
x_1 &= 1 - X_1 + X_2 - X_1X_2X_{12},\\
x_2 &= 1 + X_1 - X_2 - X_1X_2X_{12},\\
x_3 &= 1 + X_1X_{13} + X_2X_{23} + X_1X_2X_{12}X_{13}X_{23}X_{123},\\
x_{12} &= 1 - X_1 - X_2 + X_1X_2X_{12},\\
x_{13} &= 1 - X_1X_{13} + X_2X_{23} - X_1X_2X_{12}X_{13}X_{23}X_{123},\\
x_{23} &= 1 + X_1X_{13} - X_2X_{23} - X_1X_2X_{12}X_{13}X_{23}X_{123},\\
x_{123} &= 1 - X_1X_{13} - X_2X_{23} + X_1X_2X_{12}X_{13}X_{23}X_{123}
\end{aligned}$$

and

$$Z_i = x_i,\ Z_{12} = \frac{x_0 x_{12}}{x_1 x_2}, Z_{13} = \frac{x_0 x_{13}}{x_1 x_3}, Z_{23} = \frac{x_0 x_{23}}{x_2 x_3}, Z_{123} = \frac{x_1 x_2 x_3 x_{123}}{x_0 x_{12} x_{13} x_{23}}.$$

5 MacWilliams Idenity for Homogeneous Jacobi Polynomial

In this section, we give the MacWilliams identity for the homogeneous Jacobi polynomial of a code.

In order to homogenize the Jacobi polynomial, we introduce the new variables

$$X_{k_1 \dots k_\ell} = \prod_{\substack{0 \le j \le \ell \\ 1 \le i_1 < \cdots < i_j \le \ell \\ \{k_{i_1}, \dots, k_{i_j}\} \subset \{k_1, \dots, k_\ell\}}} y_{k_{i_1} \dots k_{i_j}}^{(-1)^{\ell - j}}.$$

Then the homogeneous Jacobi polynomial $\mathfrak{Jac}(C, v; \{y_a\}_{a \in \mathbf{F}_2^{g+1}})$ is defined as follows.

$$\mathfrak{Jac}^{(g)}(C, v;\ \{y_a\}_{a\in\mathbf{F}_2^{g+1}})$$

$$= y_0^n \left(\frac{y_{g+1}}{y_0}\right)^{wt(v)} Jac^{(g)}(C, v; \left\{\frac{y_i}{y_0}\right\}_{1\le i\le g}, \left\{ \prod_{\substack{0\le j\le \ell \\ 1\le i_1<\cdots<i_j\le \ell \\ \{k_{i_1},\ldots,k_{i_j}\}\subset\{k_1,\ldots,k_\ell\}}} y_{k_{i_1}\ldots k_{i_j}}^{(-1)^{\ell-j}} \right\}_{\substack{2\le \ell\le g+1 \\ 1\le k_1<\cdots<k_\ell\le g+1}})$$

$$= \sum_{u_1,\ldots,u_g\in C} y_0^n \prod_{\substack{1\le \ell\le g+1 \\ 1\le k_1<\cdots<k_\ell\le g+1}} \left(\prod_{\substack{0\le j\le \ell \\ 1\le i_1<\cdots<i_j\le \ell \\ \{k_{i_1},\ldots,k_{i_j}\}\subset\{k_1,\ldots,k_\ell\}}} y_{k_{i_1}\ldots k_{i_j}}^{(-1)^{\ell-j}} \right)^{wt(u_{k_1}\circ\cdots\circ u_{k_\ell})}$$

$$= \sum_{u_1,\ldots,u_g\in C} \prod_{a\in\mathbf{F}_2^{g+1}} y_a^{n_a(u_1\ \ldots\ u_g\ v)}$$

This is a homogeneous polynomial of total degree n. Next we give the MacWilliams identiy for $\mathfrak{Jac}(C, v;\ \{y_a\})$.

Theorem 12 *For a code C of length n, we have*

$$\mathfrak{Jac}(C^\perp, v, g : \{y_a\}_{a\in\mathbf{F}_2^{g+1}}) = \frac{1}{|C|^g}\mathfrak{Jac}(C, v; \{\sum_{\substack{b_1,\ldots,b_g\in\mathbf{F}_2 \\ b_{g+1}=a_{g+1}}} (-1)^{a_1b_1+\cdots+a_gb_g} y_{(b_1\ \ldots\ b_g\ b_{g+1})}\}_{a\in\mathbf{F}_2^{g+1}}).$$

Proof. We have

$$\begin{aligned}
\mathfrak{Jac}(C^{\perp}, v; \{y_a\}) &= \sum_{u_1,\ldots,u_g \in C^{\perp}} \prod_{a \in \mathbf{F}_2^{g+1}} y_a^{n_a(u_1,\ldots,u_g,v)} \\
&= \sum_{w_1,\ldots,w_g \in \mathbf{F}_2^n} \delta_{C^{\perp}}(w_1) \ldots \delta_{C^{\perp}}(w_g) \prod_a y_a^{n_a(w_1,\ldots,w_g,v)} \\
&= \frac{1}{|C|^g} \sum_{\substack{u_1,\ldots,u_g \in C \\ w_1,\ldots,w_g \in \mathbf{F}_2^n}} (-1)^{u_1 \cdot w_1 + \cdots + u_g \cdot w_g} \prod_a y_a^{n_a(w_1,\ldots,w_g,v)} \\
&= \frac{1}{|C|^g} \sum_{\substack{u_1,\ldots,u_g \in C \\ w_1,\ldots,w_g \in \mathbf{F}_2^n}} (-1)^{(u_{11}w_{11} + \cdots + u_{1n}w_{1n}) + \cdots + (u_{g1}w_{g1} + \cdots + u_{gn}w_{gn})} \\
&\quad \times y_{(w_{11} \ \ldots \ w_{g1} \ v_1)} \cdots y_{(w_{1n} \ \ldots \ w_{gn} \ v_n)} \\
&= \frac{1}{|C|^g} \sum_{u_1,\ldots,u_g \in C} \prod_{1 \le i \le n} \left\{ \sum_{w_{1i},\ldots,w_{gi} \in \mathbf{F}_2} (-1)^{u_{1i}w_{1i} + \cdots + u_{gi}w_{gi}} y_{(w_{1i} \ \ldots \ w_{gi} \ v_i)} \right\} \\
&= \frac{1}{|C|^g} \sum_{u_1,\ldots,u_g \in C} \prod_{a \in \mathbf{F}_2^{g+1}} \left\{ \sum_{\substack{b_1,\ldots,b_g \in \mathbf{F}_2 \\ b_{g+1} = a_{g+1}}} (-1)^{a_1 b_1 + \cdots + a_g b_g} y_{(b_1 \ \ldots \ b_g \ b_{g+1})} \right\}^{n_a(u_1,\ldots,u_g,v)} \\
&= \frac{1}{|C|^g} \mathfrak{Jac}(C, v; \{ \sum_{\substack{b_1,\ldots,b_g \in \mathbf{F}_2 \\ b_{g+1} = a_{g+1}}} (-1)^{a_1 b_1 + \cdots + a_g b_g} y_{(b_1 \ \ldots \ b_g \ b_{g+1})} \}_{a \in \mathbf{F}_2^{g+1}}).
\end{aligned}$$

This completes the proof of the MacWilliams identity for the homogeneous Jacobi polynomial.

We will discuss the applications of the MacWilliams identity in the subsequent papers.

Acknowledgment. This work was supported by JSPS KAKENHI Grant Number 17K05164. The computations were done with Magma [2] and Maple.

References

[1] Berlekamp, E.R., MacWilliams, F.J., Sloane, N.J.A., Gleason's theorem on self-dual codes, IEEE Trans. Information Theory IT-18 (1972), 409-

414.

[2] Bosma, W., Cannon, J., Playoust, C., The Magma algebra system I: The user language, J. Symbolic Comput. 24 (1997), no. 3-4, 235-265.

[3] Broué, M., Enguehard, M., Polynômes des poids de certains codes et fonctions thêta de certains réseaux, Ann. Sci. École Norm. Sup. (4) 5 (1972), 157-181.

[4] Duke, W., On codes and Siegel modular forms, Internat. Math. Res. Notices 1993, no. 5, 125-136.

[5] Eichler, M., Zagier, D., The theory of Jacobi forms, Progress in Mathematics, 55. Birkhäuser Boston, Inc., Boston, MA, 1985.

[6] Gleason, A.M., Weight polynomials of self-dual codes and the MacWilliams identities, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 3, pp. 211-215. Gauthier-Villars, Paris, 1971.

[7] Huffman, W. C., Pless, V., Fundamentals of error-correcting codes, Cambridge University Press, Cambridge, 2003.

[8] MacWilliams, F.J., Mallows, C.L., Sloane, N.J.A., Generalizations of Gleason's theorem on weight enumerators of self-dual codes, IEEE Trans. Information Theory IT-18 (1972), 794-805.

[9] MacWilliams, F. J., Sloane, N. J. A., The theory of error-correcting codes, North-Holland Mathematical Library, Vol. 16, 2003.

[10] Nebe, G., Rains, E.M., Sloane, N.J.A., Self-dual codes and invariant theory, Algorithms and Computation in Mathematics, 17. Springer-Verlag, Berlin, 2006.

[11] Oura, M., Ozeki, M., Distinguishing Siegel theta series of degree 4 for the 32-dimensional even unimodular extremal lattices, Abh. Math. Semin. Univ. Hambg. 86 (2016), no. 1, 19-53.

[12] Ozeki, M., Hadamard matrices and doubly even self-dual error-correcting codes, J. Combin. Theory Ser. A 44 (1987), no. 2, 274-287.

[13] Ozeki, M., Examples of even unimodular extremal lattices of rank 40 and their Siegel theta-series of degree 2, J. Number Theory 28 (1988), no. 2, 119-131.

[14] Ozeki, M., On the relation between the invariants of a doubly even self-dual binary code C and the invariants of the even unimodular lattices $L(C)$ defined from the code C; in Meeting on algebraic combinatorics, Suuri Kaiseki Kenkyusho Kokyuroku No.671 (1998).

[15] Ozeki, M, On the notion of Jacobi polynomials for codes, Math. Proc. Cambridge Philos. Soc. 121 (1997), no. 1, 15-30.

[16] Runge, B., Codes and Siegel modular forms, Discrete Math. 148 (1996), no. 1-3, 175-204.

[17] Yamazaki, T., Jacobi forms and a Maass relation for Eisenstein series, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 33 (1986), no. 2, 295-310.

[18] Ziegler, C., Jacobi forms of higher degree, Abh. Math. Sem. Univ. Hamburg 59 (1989), 191-224.

keita.homma12f2@gmail.com

PFU LIMITED
Japan
wondertooneself@gmail.com

Institute of Science and Engineering
Kanazawa University
Japan
oura@se.kanazawa-u.ac.jp