Certain subrings in classical invariant theory

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Abstract

We have studied E-polynomials which are combinatorial analogue of Eisenstein series. In this paper, we apply this approach to classical invariant theory. The corresponding subrings to E-polynomials are investigated.

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1 Introduction

Eisenstein series are important in number theory. Under the correspondence between combinatorics and modular forms, we have introduced the notion of E-polynomials.

On the other hand, classical invariant theory plays important roles in many branches of mathematics. Igusa [2] discussed invariant theory of binary forms and arithmetic invariants. The connection between the modular forms and projective invariants then was given in [3]. The structure of the graded ring of invariants of binary octavics then was given by Shioda in [8]. In this paper, we construct the analogue theory of Eisenstein series. We apply the notion to classical invariant theory. The computations are done by [1] and [6].

Let m be a positive integer. We take a ground form of degree m

$$f = \sum_{i=0}^{m} u_i \binom{m}{i} \xi_1^{m-i} \xi_2^{i}.$$

While $\xi_1,\ \xi_2$ are transformed according to

$$(\xi_1\ \xi_2) = (\xi_1'\ \xi_2')A\ (\text{"contragrediently"}),$$

f changes into a form of the new variables ξ_1', ξ_2' with the coefficients u_0', u_1', …, u_m' where

$$\begin{pmatrix} u_0' \\ u_1' \\ \vdots \\ u_m' \end{pmatrix} = \big(A\big)_m \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_m \end{pmatrix}.$$

We write this correspondence $u' = \big(A\big)_m u$ for short. This gives an irreducible representation $(A \mapsto \big(A\big)_m)$ of $SL(2, \mathbf{C})$ of degree $m+1$.

We let $SL(2, \mathbf{C})$ operate on $\mathbf{C}[u] = \mathbf{C}[u_0, u_1, \ldots, u_m]$ by the above representation and consider the invariant subring, say $S(2, m)$:

$$S(2, m) := \{J \in \mathbf{C}[u] : \ J(u') = J(u), \ {}^{\forall}A \in SL(2, \mathbf{C})\}.$$

It is known that $S(2, m)$ is of finite type over $\mathbf{C}$. For example, we have

$$S(2, 2) = \mathbf{C}[u_0 u_2 - u_1^2].$$

Let M be a graded ring such that each homogeneous part M_d of degree d is a finite dimensional vector space over $M_0 = \mathbf{C}$. We can write M as

$$M = \bigoplus_{d=0}^{\infty} M_d.$$

The dimension formula of M is defined by the formal series

$$\sum_{d=0}^{\infty} (\dim M_d) t^d.$$

The following formulas are the dimension formulas of $S(2, m)$ for $m = 2, 4, 6, 8$.

$$\begin{aligned}
S(2,2) &: \sum_{d=0}^{\infty} (\dim S_d(2,2)) t^d = \frac{1}{1-t^2}, \\
S(2,4) &: \sum_{d=0}^{\infty} (\dim S_d(2,4)) t^d = \frac{1}{(1-t^2)(1-t^3)}, \\
S(2,6) &: \sum_{d=0}^{\infty} (\dim S_d(2,6)) t^d = \frac{1+t^{15}}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})}, \\
S(2,8) &: \sum_{d=0}^{\infty} (\dim S_d(2,8)) t^d = \frac{1+t^8+t^9+t^{10}+t^{18}}{\prod_{i=2}^{7}(1-t^i)}.
\end{aligned}$$

The generators of the rings mentioned are known. For example, $S(2,4)$ is generated by P and Q whose explicit forms are

$$P = u_0u_4 - 4u_1u_3 + 3u_2^2,$$
$$Q = \det\begin{pmatrix} u_0 & u_1 & u_2 \\ u_1 & u_2 & u_3 \\ u_2 & u_3 & u_4 \end{pmatrix}.$$

The ring $S(2,6)$ are generated by 5 elements J_2, J_4, J_6, J_{10}, and J_{15}. We write the generators of $S(2,6)$ in Appendix A. In this paper, we deal with only invariants of even degrees. So we omit J_{15}. We denote by $S(2,m)^e$ the even parts of $S(2,m)$.

In order to obtain the useful construction of invariants, we shall interpret the ground form as

$$f = u_0 \prod_{i=1}^{m} (\xi_1 - \varepsilon_i\xi_2).$$

As usual, we denote by S_n the symmetric group of degree n. The following lemma gives a construction of invariants we expected (*cf.* [2]).

Lemma 1. *An expression of the form*

$$u_0^r \sum (\varepsilon_i - \varepsilon_j)(\varepsilon_k - \varepsilon_l)\ldots,$$

in which every ε_i appears r times in each product and which is symmetric in $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ can be considered as an invariant of degree r.

2 Results

Let g be a positive integer. We start with a ground form of degree $2g+2$

$$\begin{aligned} f &= \sum_{i=0}^{2g+2} u_i \binom{2g+2}{i} \xi_1^{(2g+2)-i}\xi_2^i \\ &= u_0 \prod_{i=1}^{2g+2} (\xi_1 - \varepsilon_i\xi_2). \end{aligned}$$

We would like to concentrate on one type of invariants we shall define now. We fix the following polynomial

$$\varphi_{2n} = u_0^{2n}(\varepsilon_1 - \varepsilon_2)^{2n}(\varepsilon_3 - \varepsilon_4)^{2n}\ldots(\varepsilon_{2g+1} - \varepsilon_{2g+2})^{2n}.$$

We denote by G the symmetric group of degree $2g+2$. The group G acts on the polynomial ring $\mathbf{C}[\varepsilon_1, \dots, \varepsilon_{2g+2}]$ as $F(\dots, \varepsilon_i, \dots)^\sigma = F(\dots, \varepsilon_{i^\sigma}, \dots)$. Let $G_{\varphi_{2n}}$ be the stabilizer of φ_{2n}, that is, elements of G which preserve φ_{2n}.

Proposition 2. *The group $G_{\varphi_{2n}}$ can be generated by the $(g+1)+2$ elements*

$$(1\ 2), (3\ 4), \dots, (2g+1\ 2g+2),$$
$$(1\ 3)(2\ 4), (1\ 3\ 5\ \dots\ 2g+1)(2\ 4\ \dots\ 2g+2)$$

and is isomorphic to $C_2^{g+1} \rtimes S_{g+1}$. In particular, $G_{\varphi_{2n}}$ does not depend on n.

Proof. The elements given in Proposition 2 are in $G_{\varphi_{2n}}$. Conversely, since $(\varepsilon_i - \varepsilon_j)^{2n} = (\varepsilon_j - \varepsilon_i)^{2n}$, the first $g+1$ elements can be obtained by interchanging of two indexes in each parenthesis. These interchanging are isomorphic to C_2^{g+1}. Let $\tilde{1}, \tilde{2}, \dots, \widetilde{g+1}$ represent $(1\ 2), (3\ 4), \dots, (g+1\ g+2)$, respectively. Then, the additional two generators come from the generators of the set of all permutations of $\{\tilde{1}, \tilde{2}, \dots, \widetilde{g+1}\}$. This set is isomorphic to S_{g+1} and its generators are $(\tilde{1}\ \tilde{2})$ and $(\tilde{1}\ \dots\ \widetilde{g+1})$ which represent $(1\ 2)(3\ 4)$ and $(1\ 3\ \dots\ 2g+1)(2\ 4\ \dots\ 2g+2)$, respectively. □

For simplicity, we denote by K for $G_{\varphi_{2n}}$ and by κ the cardinality of $K \backslash G$. The number κ for $g = 1, 2, 3$ is 3, 15, 105, respectively.

Set

$$\psi_{2n} = \sum_{K \backslash G \ni \sigma} \varphi_{2n}^\sigma,$$

which is actually an element of degree $2n$ in $S(2, 2g+2)$ by Lemma 1. We call the polynomial ψ_{2n} by an *E-polynomial*. We shall denote by A_g the ring generated by ψ_{2n} $(n = 1, 2, \dots)$ over $\mathbf{C}$. The ring A_g is a subring of the invariant ring $S(2, 2g+2)$.

Theorem 3. *The ring A_g is finitely generated over $\mathbf{C}$. More precisely the elements $\psi_2, \psi_4, \dots, \psi_{2\kappa}$ generate the ring A_g.*

Proof. Since the second assertion implies the first, we shall show the second. Let $\sigma_1, \sigma_2, \dots, \sigma_\kappa$ be a set of representatives of $K \backslash G$. For each σ_i, the polynomial $\varphi_{2n}^{\sigma_i}$ can be written as

$$\varphi_{2n}^{\sigma_i} = B_i^{2n}$$

where

$$B_i = \left(u_0 \left(\varepsilon_1 - \varepsilon_2\right) \left(\varepsilon_3 - \varepsilon_4\right) \dots \left(\varepsilon_{2g+1} - \varepsilon_{2g+2}\right)\right)^{\sigma_i}.$$

Because of the last statement of Proposition 2, the polynomial ψ_{2n} has the form

$$\psi_{2n} = B_1^{2n} + B_2^{2n} + \cdots + B_\kappa^{2n}$$

for $n = 1, 2, \ldots$. By applying the fundamental theorem of symmetric functions to our situation, we see that A_g is generated by $\psi_2, \psi_4, \ldots, \psi_{2\kappa}$. $\square$

The natural question arising from Theorem 3 is if we can find the minimal generators of A_g. On this point, we have the following theorem.

Theorem 4. *(1) A_1 is generated by ψ_2, ψ_6 and coincides with $S(2,4)^e$.*
(2) A_2 is generated by $\psi_2, \psi_4, \psi_6, \psi_{10}$ and coincides with $S(2,6)^e$.
(3) A_3 is strictly smaller than $S(2,8)^e$.

Proof. We prove Theorem 4 by showing the relationship with the known generators. Starting from $g = 1$, the polynomials ψ_2 and ψ_6 can be expressed in P and Q as

$$\begin{aligned}
\psi_2 &= 24 \cdot P, \\
\psi_6 &= 2^7 \cdot 3 \cdot 11 \cdot P^3 - 2^8 \cdot 3^4 \cdot Q^2.
\end{aligned}$$

Now we continue for $g = 2$. By J_2, J_4, J_6, and J_{10}, the polynomials ψ_2, ψ_4, ψ_6, and ψ_{10} can be expressed as

$$\begin{aligned}
\psi_2 =\ & -2^4 \cdot 3 \cdot 5 \cdot J_2, \\
\psi_4 =\ & 2^3 \cdot 3 \cdot 5 \cdot 71 \cdot J_2^2 + 2^5 \cdot 3^3 \cdot 5^3 \cdot J_4, \\
\psi_6 =\ & -2^5 \cdot 3^3 \cdot 5 \cdot 7^2 \cdot J_2^3 - 2^7 \cdot 3^5 \cdot 5^3 \cdot 7 J_2 J_4 + 2^3 \cdot 3 \cdot 5^4 \cdot 13 \cdot J_6, \\
\psi_{10} =\ & 2^5 \cdot 3^3 \cdot 5 \cdot 17 \cdot 15287 \cdot J_2^5 - 2^8 \cdot 3^5 \cdot 5^4 \cdot 29 \cdot 199 \cdot J_2^3 J_4 \\
& + 2^3 \cdot 3 \cdot 5^5 \cdot 37 \cdot 857 J_2^2 J_6 - 2^9 \cdot 3^7 \cdot 5^6 \cdot 229 J_2 J_4^2 \\
& + 2^5 \cdot 3^3 \cdot 5^6 \cdot 2207 \cdot J_4 J_6 - 2^5 \cdot 3^6 \cdot 5^6 \cdot 31 \cdot J_{10}.
\end{aligned}$$

For $g = 3$, the dimension formula of $S(2,8)$ is

$$\begin{aligned}
\sum_{d=0}^{\infty} (\dim S_d(2,8)) t^d &= \frac{1 + t^8 + t^9 + t^{10} + t^{18}}{\prod_{i=2}^{7}(1 - t^i)} \\
&= 1 + t^2 + t^3 + 2t^4 + 2t^5 + 4t^6 + 4t^7 + 7t^8 + \cdots .
\end{aligned}$$

The dimension of $S(2,8)$ of degree 8 is 7. However, the dimension of A_3 of degree 8 is at most 5. $\square$

For the comparation of the dimension formula, we give an example for $g = 2$. The dimension formula of $S(2, 6)$ is

$$\frac{1 + t^{15}}{(1 - t^2)(1 - t^4)(1 - t^6)(1 - t^{10})},$$

while the dimension formula of A_2 is

$$\frac{1}{(1 - t^2)(1 - t^4)(1 - t^6)(1 - t^{10})}.$$

In the paper [5], the relation between the ring $S(2, 2g+2)$ and the weight enumerators of some codes was discussed. In Appendix B, we give the relations between the weight enumerators and E-polynomials for $g = 1, 2$.

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Appendix A Generators of $S(2,6)$

These are the generators of $S(2,6)$ taken from [7].

$$
\begin{aligned}
J_2 &= u_0u_6 - 6u_1u_5 + 15u_2u_4 - 10u_3^2,\\
J_4 &= \det \begin{pmatrix} u_0 & u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 & u_4 \\ u_2 & u_3 & u_4 & u_5 \\ u_3 & u_4 & u_5 & u_6 \end{pmatrix},\\
J_6 &= \det \begin{pmatrix} b_0 & b_1 & b_2 \\ b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 \end{pmatrix},\\
J_{10} &= u_0\ c^3 - 6u_1\ bc^2 + 3u_2(ac+4b^2)c - 4u_3(3abc+2b^3) + 3u_4\ a(ac+4b^2)\\
&\quad - 6u_5\ a^2b + u_6\ a^3,
\end{aligned}
$$

where

$$
\begin{aligned}
b_0 &= 6(u_0u_4 - 4u_1u_3 + 3u_2^2),\\
b_1 &= 3(u_0u_5 - 3u_1u_4 + 2u_2u_3),\\
b_2 &= u_0u_6 - 9u_2u_4 + 8u_3^2,\\
b_3 &= 3(u_1u_6 - 3u_2u_5 + 2u_3u_4),\\
b_4 &= 6(u_2u_6 - 4u_3u_5 + 3u_4^2),\\
a &= 2(u_0u_2u_6 - 3u_0u_3u_5 + 2u_0u_4^2 - u_1^2u_6 + 3u_1u_2u_5 - u_1u_3u_4 - 3u_2^2u_4 + 2u_2u_3^2),\\
b &= u_0u_3u_6 - u_0u_4u_5 - u_1u_2u_6 - 8u_1u_3u_5 + 9u_1u_4^2 + 9u_2^2u_5 - 17u_2u_3u_4 + 8u_3^3,\\
c &= 2(u_0u_4u_6 - u_0u_5^2 - 3u_1u_3u_6 + 3u_1u_4u_5 + 2u_2^2u_6 - u_2u_3u_5 - 3u_2u_4^2 + 2u_3^2u_4).
\end{aligned}
$$

Appendix B Weight Enumerator

We recall coding theory. A code C of length n means a subspace of $\mathbb{F}_2^n$. The weight $wt(x)$ of $x \in \mathbb{F}_2^n$ means the number of nonzero x_i. The inner product of two elements $x, y \in \mathbb{F}_2^n$ is defined by

$$x \cdot y := \sum x_i\, y_i \in \mathbb{F}_2.$$

The dual code $C^{\perp}$ of C is defined by the subspace of $\mathbb{F}_2^n$ whose elements are orthogonal to every element of C. If $C = C^{\perp}$, then we call C self-dual. The code C is called *doubly even* if the weight of any element in C is equivalent to 0 modulo 4. The weight enumerator of C in genus g is defined by

$$W_C^{(g)} := W_C^{(g)}(x_a | a \in \mathbb{F}_2^g) = \sum_{c_1,\ldots,c_g \in C} \prod_{a \in \mathbb{F}_2^g} x_a^{n_a(c_1,\ldots,c_g)}$$

where

$$n_a(c_1, \ldots, c_g) = |\{i \,|\, (c_{1i}, \ldots, c_{gi}) = a\}|.$$

Let $\widetilde{\rho}$ be the combination of the Broué-Enguehard map Th and Igusa's homomorphism ρ. In other word, we can say

$$\widetilde{\rho}(W_C^{(g)}) = \rho(Th(W_C^{(g)}))$$

for a code C. We omit the detail of $\widetilde{\rho}$ and only say that $\widetilde{\rho}$ maps the weight enumerators in genus g to the ring $S(2, 2g+2)$. The reader who is interested in the detail of $\widetilde{\rho}$ can refer to [5]. For every code C used here, the expression of $\widetilde{\rho}(W_C^{(g)})$ is taken from [5].

We start with $g = 1$. The weight enumerators of some codes are related to E-polynomials by the following relations.

$$\widetilde{\rho}(W_{e_8}^{(1)}) = 2^{-1}\,\psi_2,$$
$$\widetilde{\rho}(W_{g_{24}}^{(1)}) = 2^{-5} \cdot 11 \cdot \psi_2^3 - 2^{-3} \cdot 7\psi_6.$$

For $g = 2$, the relation between the weight enumerators and E-polynomials are the following.

$$
\begin{aligned}
\widetilde{\rho}(W_{e_8}^{(2)}) &= 2^{-4}\psi_2^2 - 3\cdot 2^{-3}\psi_4\\
\widetilde{\rho}(W_{g_{24}}^{(2)}) &= 2^{-15}\cdot 3^2\cdot 5^{-1}\cdot 13^{-2}\cdot 31^{-1}\cdot 20129\psi_2^6 - 2^{-14}\cdot 5\cdot 13^{-2}\cdot 31^{-1}\cdot 59651\psi_2^4\psi_4\\
&\quad + 2^{-13}\cdot 7\cdot 13^{-2}\cdot 31^{-1}\cdot 809\psi_2^3\psi_6 - 2^{-9}\cdot 3\cdot 5^{-1}\cdot 7\cdot 11\cdot 31^{-1}\psi_2\psi_{10}\\
&\quad + 2^{-11}\cdot 3\cdot 13^{-2}\cdot 31^{-1}\cdot 65287\psi_2^2\psi_4^2 + 2^{-12}\cdot 7\cdot 13^{-2}\cdot 29\cdot 31^{-1}\cdot 149\psi_2\psi_4\psi_6\\
&\quad - 2^{-10}\cdot 3\cdot 11\psi_4^3 + 2^{-7}\cdot 3^2\cdot 7\cdot 13^{-2}\psi_6^2\\
\widetilde{\rho}(W_{d_{24}^+}^{(2)}) &= 2^{-10}\cdot 5^{-1}\cdot 13^{-2}\cdot 31^{-1}\cdot 6323\psi_2^6 - 2^{-9}\cdot 13^{-2}\cdot 31^{-1}\cdot 12143\psi_2^4\psi_4\\
&\quad - 2^{-10}\cdot 3\cdot 13^{-2}\cdot 31^{-1}\cdot 683\psi_2^3\psi_6 + 2^{-6}\cdot 3\cdot 5^{-1}\cdot 11\cdot 31^{-1}\psi_2\psi_{10}\\
&\quad + 2^{-10}\cdot 3^2\cdot 13^{-2}\cdot 31^{-1}\cdot 47\cdot 379\psi_2^2\psi_4^2 - 2^{-9}\cdot 13^{-2}\cdot 31^{-1}\cdot 2089\psi_2\psi_4\psi_6\\
&\quad - 2^{-9}\cdot 3\cdot 11\psi_4^3 - 2^{-5}\cdot 3^2\cdot 13^{-2}\psi_6^2\\
\widetilde{\rho}(W_{d_{32}^+}^{(2)}) &= 2^{-16}\cdot 5^{-1}\cdot 13^{-2}\cdot 31^{-1}\cdot 20507\psi_2^8 - 2^{-13}\cdot 3^{-1}\cdot 7\cdot 13^{-2}\cdot 23\cdot 31^{-1}\cdot 271\psi_2^6\psi_4\\
&\quad - 2^{-11}\cdot 5^{-1}\cdot 13^{-2}\cdot 23\cdot 31^{-1}\cdot 227\psi_2^5\psi_6 + 2^{-13}\cdot 3^{-1}\cdot 13^{-2}\cdot 15541\psi_2^4\psi_4^2\\
&\quad + 2^{-9}\cdot 3^{-1}\cdot 13^{-2}\cdot 31^{-1}\cdot 4679\psi_2^3\psi_4\psi_6 + 2^{-7}\cdot 5^{-1}\cdot 13^{-1}\cdot 31^{-1}\cdot 173\psi_2^3\psi_{10}\\
&\quad - 2^{-11}\cdot 7\cdot 13^{-2}\cdot 31^{-1}\cdot 27743\psi_2^2\psi_4^3 - 2^{-6}\cdot 13^{-2}\cdot 31^{-1}\cdot 139\psi_2^2\psi_6^2\\
&\quad + 2^{-9}\cdot 3^{-1}\cdot 13^{-2}\cdot 31^{-1}\cdot 2129\psi_2\psi_4^2\psi_6 - 2^{-6}\cdot 13^{-1}\cdot 31^{-1}\cdot 107\psi_2\psi_4\psi_{10}\\
&\quad + 2^{-12}\cdot 3\cdot 43\psi_4^4 + 2^{-5}\cdot 13^{-2}\cdot 31^{-1}\cdot 281\psi_4\psi_6^2 + 2^{-1}\cdot 3\cdot 5^{-1}\cdot 13^{-1}\cdot 31^{-1}\psi_6\psi_{10}\\
\widetilde{\rho}(W_{d_{40}^+}^{(2)}) &= 2^{-18}\cdot 3^{-1}\cdot 5^{-1}\cdot 13^{-2}\cdot 31^{-2}\cdot 267941\psi_2^{10} - 2^{-16}\cdot 3^{-1}\cdot 13^{-2}\cdot 31^{-2}\cdot 606959\,\psi_2^8\psi_4\\
&\quad - 2^{-18}\cdot 3^{-1}\cdot 13^{-2}\cdot 31^{-2}\cdot 1877033\,\psi_2^7\psi_6 + 2^{-18}\cdot 3^{-1}\cdot 13^{-2}\cdot 31^{-2}\cdot 281^2\cdot 541\,\psi_2^6\psi_4^2\\
&\quad + 2^{-14}\cdot 5^{-1}\cdot 13^{-1}\cdot 17\cdot 31^{-2}\cdot 4871\,\psi_2^5\psi_{10} + 2^{-17}\cdot 3^{-1}\cdot 13^{-2}\cdot 31^{-2}\cdot 2207\cdot 5779\,\psi_2^5\psi_4\psi_6\\
&\quad - 2^{-17}\cdot 5\cdot 13^{-1}\cdot 31^{-2}\cdot 903827\,\psi_2^4\psi_4^3 - 2^{-14}\cdot 3^{-1}\cdot 5\cdot 13^{-2}\cdot 31^{-2}\cdot 107209\,\psi_2^4\psi_6^2\\
&\quad - 2^{-16}\cdot 3^{-1}\cdot 5\cdot 13^{-2}\cdot 17\cdot 31^{-2}\cdot 59\cdot 4957\psi_2^3\psi_4^2\psi_6 - 2^{-12}\cdot 7\cdot 13^{-1}\cdot 31^{-2}\cdot 7187\,\psi_2^3\psi_4\psi_{10}\\
&\quad + 2^{-16}\cdot 3\cdot 5\cdot 13^{-2}\cdot 31^{-2}\cdot 37\cdot 205187\psi_2^2\psi_4^4 + 2^{-12}\cdot 3^{-1}\cdot 5\cdot 13^{-2}\cdot 31^{-2}\cdot 271919\,\psi_2^2\psi_4\psi_6^2\\
&\quad + 2^{-9}\cdot 13^{-1}\cdot 17\cdot 31^{-2}\cdot 43\psi_2^2\psi_6\psi_{10} + 2^{-15}\cdot 5\cdot 7\cdot 13^{-2}\cdot 31^{-2}\cdot 79319\psi_2\psi_4^3\psi_6\\
&\quad + 2^{-12}\cdot 3\cdot 13^{-1}\cdot 31^{-2}\cdot 181\cdot 293\,\psi_2\psi_4^2\psi_{10} - 2^{-15}\cdot 3^3\cdot 19\,\psi_4^5\\
&\quad - 2^{-12}\cdot 3^{-1}\cdot 5\cdot 13^{-2}\cdot 31^{-2}\cdot 71\cdot 6719\psi_4^2\psi_6^2 - 2^{-8}\cdot 13^{-1}\cdot 17\cdot 31^{-2}\cdot 293\psi_4\psi_6\psi_{10}\\
&\quad + 2^{-6}\cdot 3\cdot 5^{-1}\cdot 31^{-2}\cdot 41\psi_{10}^2.
\end{aligned}
$$