Weight enumerator, intersection enumerator and Jacobi polynomial

K.Honma, M.Oura

Abstract

The intersection enumerator and the Jacobi polynomial in *arbitrary* genus for a binary code are introduced. Adding the weight enumerator into our discussion, we give the explicit relations among them and describe some basic properties.

1 Introduction

The weight enumerator plays an important role in coding theory. Gleason [6] initiated application of the weight enumerator to the invariant theory of the finite groups and soon after that, Broué-Enguehard [3] constructed a modular form from the weight enumerator. These works were generalized to higher genus [1, 7, 4, 14, 8]. In [13], Ozeki gave the new notion "Jacobi polynomial" of a code. Ozeki commented that this comes out of considerations on various invariants of codes [10, 11, 12] and on Jacobi theta-series [5]. In [9], the notion of the intersection enumerator is given for some computations of extremal codes. In the present paper, we discuss these polynomials in *arbitrary* genus with future applications in mind and some results in [13] are generalized to the case in higher genus.

We shall recall coding theory. In the present paper we restrict to the binary case. Let $\mathbf{F}_2 = \{0, 1\}$ be the field of two elements and $\mathbf{F}_2^n$ the n-dimensional vector space over $\mathbf{F}_2$ equipped with the usual inner product

$$u \cdot v = u_1 v_1 + \cdots + u_n v_n,\ u = (u_1, \ldots, u_n),\ v = (v_1, \ldots, v_n) \in \mathbf{F}_2^n.$$

Keywords: code, weight enumerator, intersection enumerator, Jacobi polynomial
2010MSC: Primary 94B05, Secondary 05E15
Running Head: Weight enumerator, intersection enumerator and Jacobi polynomial

For $v_1 = (v_{11}, \ldots, v_{1n_1}) \in \mathbf{F}_2^{n_1},\ v_2 = (v_{21}, \ldots, v_{2n_2}) \in \mathbf{F}_2^{n_2}$, we put

$$v_1 \oplus v_2 = (v_{11}, \ldots, v_{1n_1}, v_{21}, \ldots, v_{2n_2}).$$

We introduce the operation $\circ$ on $\mathbf{F}_2^n$ as

$$u \circ v = (u_1 v_1, \ldots, u_n v_n).$$

This satisfies the associativity $(u_1 \circ u_2) \circ u_3 = u_1 \circ (u_2 \circ u_3)$. We denote by $e_{i_1 \ldots i_\ell}$ the element of $\mathbf{F}_2^g$, whose entry is 1 for the i_1-,..., i_ℓ-part, 0 otherwise. For example, $e_{12} = (1,1,0) \in \mathbf{F}_2^3$. Therefore every non-zero element of $\mathbf{F}_2^g$ can be expressed as $e_{i_1 \ldots i_\ell}$ for suitable $i_1, \ldots, i_\ell$. The zero vector in $\mathbf{F}_2^g$ is denoted by e_0. The weight $wt(u)$ is the number of non-zero coordinates of u. The intersection number $u * v$ in the sense of [13] is $wt(u \circ v)$ in this paper. We denote by $n_a(u_1, \ldots, u_g)$ the number of i such that $a = (u_{1i}, \ldots, u_{gi})$ for $a \in \mathbf{F}_2^g$.

A linear code of length n is a linear subspace of $\mathbf{F}_2^n$. We denote by $C^\perp$ the dual code of C

$$C^\perp = \{u \in \mathbf{F}_2^n |\ u \cdot v = 0,\ \forall v \in C\}.$$

If $C = C^\perp$, then it is called self-dual. If $wt(u) \equiv 0 \pmod 4$, $\forall u \in C$, then it is called doubly even. It is known that a self-dual and doubly even code of length n exists if and only if $n \equiv 0 \pmod 8$.

For a code C_1 (resp. C_2) of length n_1 (resp. n_2), we denote by $C_1 \oplus C_2$ the direct sum of C_1 and C_2. In other words,

$$C_1 \oplus C_2 = \{u_1 \oplus u_2 :\ u_1 \in C_1,\ u_2 \in C_2\}.$$

The (homogeneous) weight enumerator of a code C of length n is

$$W_C(x, y) = \sum_{u \in C} x^{n - wt(u)} y^{wt(u)}.$$

This is nothing but the 1st weight enumerator in the sense of the next section. The inhomogeneous weight enumerator of a code C of length n is[1] $W_C(X) = W_C(x \leftarrow 1, y \leftarrow X)$.

[1]The notation $x \leftarrow X$ means to substitute X for x.

2 Definitions of Polynomials of Codes

We start with giving the definitions of the three kinds of the polynomials. It should be emphasized that the notion of the genus attaches to each polynomial.

Definition 1 *Let g be a positive integer and C a code of length n.*

(1) *The g-th weight enumerator of C is*

$$W_C^{(g)}(\{x_a\}_{a\in\mathbf{F}_2^g}) = \sum_{u_1,\ldots,u_g\in C}\prod_{a\in\mathbf{F}_2^g} x_a^{n_a(u_1,\ldots,u_g)}.$$

(2) *The g-th Jacobi polynomial of C with the reference vector $v\in\mathbf{F}_2^n$ is*

$$Jac^{(g)}(C,v|\ \{X_i\}_{1\le i\le g}, \{X_{k_1\ldots k_\ell}\}_{\substack{2\le\ell\le g+1\\1\le k_1<\cdots<k_\ell\le g+1}})$$

$$= \sum_{u_1,\ldots,u_g\in C}\left(\prod_{1\le i\le g} X_i^{wt(u_i)}\right)\left\{\prod_{2\le\ell\le g+1}\left(\prod_{\substack{1\le k_1<\cdots<k_\ell\le g+1\\u_{g+1}=v}} X_{k_1\ldots k_\ell}^{wt(u_{k_1}\circ\cdots\circ u_{k_\ell})}\right)\right\}.$$

(3) *The g-th intersection enumerator of C is*

$$I_C^{(g)}(\{X_{k_1\ldots k_\ell}\}_{\substack{1\le\ell\le g\\1\le k_1<\cdots<k_\ell\le g}}) = \sum_{u_1,\ldots,u_g\in C}\left\{\prod_{1\le\ell\le g}\left(\prod_{1\le k_1<\cdots<k_\ell\le g} X_{k_1\ldots k_\ell}^{wt(u_{k_1}\circ\cdots\circ u_{k_\ell})}\right)\right\}.$$

When no confusion occurs, we omit the notation of variables in the polynomials as $W_C^{(g)}$, $Jac^{(g)}(C,v)$ and $I_C^{(g)}$. It is well known that the g-th weight enumerator is attracted by number theory (*cf.* [3, 4, 14]).

Remark 2 (1) $W_C^{(1)}(x_0\leftarrow 1, x_1\leftarrow X) = Jac^{(1)}(C,0|\ X_1\leftarrow X, X_{12}) = I_C^{(1)}(X_1\leftarrow X)$.

(2) *The number of variables in each polynomial is*

$W_C^{(g)}$: 2^g,

$Jac^{(g)}(C,v)$: $g+\binom{g+1}{2}+\binom{g+1}{3}+\cdots+\binom{g+1}{g+1} = 2^{g+1}-2$,

$I_C^{(g)}$: $\binom{g}{1}+\binom{g}{2}+\cdots+\binom{g}{g} = 2^g-1$.

At this point, the difference between $Jac^{(g)}(C,v)$ and $I_C^{(g+1)}$ is that $I_C^{(g+1)}$ contains X_{g+1} in the definition, whereas $Jac^{(g)}(C,v)$ does not.

(3) $I_C^{(g)}(X_{i_1\dots i_\ell} \leftarrow 1$ *for* $\ell \geq 3)$ *is the intersection enumerator in genus* g *in the sense of* [9].

We shall describe some examples. For $g = 1$, we have

$$Jac^{(1)}(C, v|\ X_1 \leftarrow X, X_{12} \leftarrow Z) = \sum_{u \in C} X^{wt(u)} Z^{wt(u \circ v)}.$$

This is the Jacobi polynomial of binary codes dealt in [13]. For $g = 2$, we have

$$Jac^{(2)}(C, v) = \sum_{u_1, u_2 \in C} X_1^{wt(u_1)} X_2^{wt(u_2)} X_{12}^{wt(u_1 \circ u_2)} X_{13}^{wt(u_1 \circ u_3)} X_{23}^{wt(u_2 \circ u_3)} X_{123}^{wt(u_1 \circ u_2 \circ u_3)}$$

where $u_3 = v$ in the right-hand side.

3 Results

The Jacobi polynomial $Jac^{(g)}(C, v)$ has the following expansion[2].

$$\begin{aligned} Jac^{(g)}(C, v) = \sum_{\substack{\{m_i\} \\ \{r_{k_1 \dots k_\ell}\}}} & b(\{m_i\}_{1 \leq i \leq g}, \{r_{k_1 \dots k_\ell}\}_{\substack{2 \leq \ell \leq g+1 \\ 1 \leq k_1 < \cdots < k_\ell \leq g+1}}) \left(\prod_{1 \leq i \leq g} X_i^{m_i} \right) \\ & \times \left\{ \prod_{2 \leq \ell \leq g+1} \left(\prod_{1 \leq k_1 < \cdots < k_\ell \leq g+1} X_{k_1 \dots k_\ell}^{r_{k_1 \dots k_\ell}} \right) \right\} \end{aligned}$$

where $b(\{m_i\}_{1 \leq i \leq g}, \{r_{k_1 \dots k_\ell}\}_{\substack{2 \leq \ell \leq g+1 \\ 1 \leq k_1 < \cdots < k_\ell \leq g+1}})$ is the number of $(u_1, \dots, u_g) \in C^g$ satisfying

$$\begin{cases} wt(u_i) = m_i & (1 \leq i \leq g), \\ wt(u_{k_1} \circ \cdots \circ u_{k_\ell}) = r_{k_1 \dots k_\ell} & (2 \leq \ell \leq g+1,\ 1 \leq k_1 < \cdots < k_\ell \leq g+1). \end{cases}$$

Here we are regarding as $u_{g+1} = v$. We have a trivial inequality $wt(u_{j_1} \circ \cdots \circ u_{j_{\ell+1}}) \leq wt(u_{i_1} \circ \cdots \circ u_{i_\ell})$ for $\{i_1, \dots, i_\ell\} \subset \{j_1, \dots, j_\ell, j_{\ell+1}\}$. From this observation, we have

[2]If we use the notation $r_i = wt(u_i)$ instead of m_i, some parts of descriptions below might be simplified, however, we did not take that way. This is because we need to emphasize the distinction between m_* and r_* and also because we respect the usage of the notation in Ozeki's original paper [13].

Proposition 3 *f there exists some pair* $\{i_1, \ldots, i_\ell\} \subset \{j_1, \ldots, j_\ell, j_{\ell+1}\}$ *for some* $\ell \geq 1$ *such that*

$$m_{i_1} < r_{j_1 j_2} \text{ for } \ell = 1$$

or

$$r_{i_1 \ldots i_\ell} < r_{j_1 \ldots j_\ell j_{\ell+1}} \text{ for } \ell \geq 2,$$

then we have $b(\{m_i\}, \{r_{i_1 \ldots i_\ell}\}) = 0$.

Here we are assuming the convention $u_{g+1} = v, \; r_{g+1} = wt(u_{g+1})$.

Proposition 4 *Let* C *be a code of length* n *which contains all one vector* 1 *and* $Jac^{(g)}(C, v) = \sum b(\{m_i\}, \{r_{k_1 \ldots k_\ell}\}) \prod X_i^{m_i} \prod X_{k_1 \ldots k_\ell}^{r_{k_1 \ldots k_\ell}}$ *the* g*-th Jacobi polynomial of* C *with the reference vector* v *of weight* k*. Fix* j $(1 \leq j \leq g)$*. Then it holds*

$$b(\{m_i\}, \{r_{k_1 \ldots k_\ell}\}) = b(\{m'_i\}, \{r'_{k_1 \ldots k_\ell}\})$$

where

$$m'_i = \begin{cases} m_i & \text{if } i \neq j, \\ n - m_j & \text{otherwise,} \end{cases}$$

and

$$r'_{k_1 \ldots k_\ell} = \begin{cases} r_{k_1 \ldots k_\ell} & \text{if } i \notin \{k_1, \ldots, k_\ell\}, \\ r_{k_1 \ldots \hat{i} \ldots k_\ell} - r_{k_1 \ldots i \ldots k_\ell} & \text{otherwise,} \end{cases}$$

and $\hat{i}$ *means to exclude* i.

Proof. For the map $(u_1, \ldots, u_i, \ldots, u_g) \mapsto (u_1, \ldots, 1 - u_i, \ldots, u_g)$ is a bijection from $\{(u_1, \ldots, u_g) \in C^g : \; wt(u_{i_1} \circ \cdots \circ u_{i_\ell}) = r_{k_1 \ldots k_\ell}, \forall k_1, \ldots, k_\ell\}$ to $\{(u_1, \ldots, u_g) \in C^g : \; wt(u_{i_1} \circ \cdots \circ u_{i_\ell}) = r'_{k_1 \ldots k_\ell}, \forall k_1, \ldots, k_\ell\}$. This completes the proof of Proposition 4.

Making successive use of this proposition, we get, for $g = 2$,

$$\begin{aligned}
& b(\{m_1, m_2\}, \{r_{12}, r_{13}, r_{23}, r_{123}\}) \\
&= b(\{n - m_1, m_2\}, \{m_2 - r_{12}, k - r_{13}, r_{23}, r_{23} - r_{123}\}) \\
&= b(\{m_1, n - m_2\}, \{m_1 - r_{12}, r_{13}, k - r_{23}, r_{13} - r_{123}\}) \\
&= b(\{n - m_1, n - m_2\}, \{n - m_1 - m_2 + r_{12}, k - r_{13}, k - r_{23}, k - r_{13} - r_{23} + r_{123}\}).
\end{aligned}$$

Next we shall consider the reduction of the intersection enumerator to the inhomogeneous weight enumerator.

Proposition 5 (1) $I_C^{(g+1)}(X_{k_1 \ldots k_\ell} \leftarrow 1 \text{ } for \ \ell \geq 2) = \prod_{i=1}^{g+1} W_C(X_i)$.

(2) $I_C^{(g+1)}(variables \ \leftarrow 1 \ except \ X_i) = |C|^g W_C(X_i)$.

Proof. For (1), we have

$$\begin{aligned}
I_C^{(g+1)}(X_{k_1 \ldots k_\ell} \leftarrow 1 \text{ for } \ell \geq 2) &= \sum_{u_1,\ldots,u_{g+1} \in C^{g+1}} \left(\prod_{1 \leq i \leq g+1} X_i^{wt(u_i)} \right) \left\{ \prod_{2 \leq \ell \leq g+1} \left(\prod_{1 \leq k_1 < \cdots < k_\ell \leq g+1} 1 \right) \right\} \\
&= \sum_{u_1,\ldots,u_{g+1} \in C^{g+1}} \left(\prod_{1 \leq i \leq g+1} X_i^{wt(u_i)} \right) \\
&= \prod_{1 \leq i \leq g+1} \left(\sum_{u_i \in C} X_i^{wt(u_i)} \right) \\
&= \prod_{1 \leq i \leq g+1} W_C(X_i).
\end{aligned}$$

The assertion (2) follows from (1) by specializing variables. This completes the proof of Proposition 5.

We would like to clarify the relationship between the weight enumerator and the intersection enumerator. In order to do so, we need the following

Lemma 6 *Let $u_1, \ldots, u_g$ be elements of $\mathbf{F}_2^n$. Then the following hold for $1 \leq \ell \leq g$.*

(1) $wt(u_{i_1} \circ \cdots \circ u_{i_\ell}) = \displaystyle\sum_{\substack{\{i_1,\ldots,i_\ell\} \subset \{j_1,\ldots,j_k\} \\ 1 \leq j_1 < \cdots < j_k \leq g}} n_{e_{j_1 \cdots j_k}}(u_1, \ldots, u_g)$.

(2) $n_{e_{i_1 \ldots i_\ell}}(u_1, \ldots, u_g) = \displaystyle\sum_{\substack{\{i_1,\ldots,i_\ell\} \subset \{j_1,\ldots,j_k\} \\ 1 \leq j_1 < \cdots < j_k \leq g}} (-1)^{k-\ell} wt(u_{j_1} \circ \cdots \circ u_{j_k})$.

Proof. (1) The right-hand side counts the coordinate positions m such that

$$u_{i_1,m} = u_{i_2,m} = \cdots = u_{i_\ell,m} = 1.$$

This is $wt(u_{i_1} \circ \cdots \circ u_{i_\ell})$.

(2) The right-hand side is

$$\sum_{\substack{\{i_1,\dots,i_\ell\}\subset\{j_1,\dots,j_k\}\\ 1\le j_1<\dots<j_k\le g}} (-1)^{k-\ell}\left\{\sum_{\substack{\{j_1,\dots,j_k\}\subset\{m_1,\dots,m_s\}\\ 1\le m_1<\dots<m_s\le g}} n_{e_{m_1\dots m_s}}(u_1,\dots,u_g)\right\}.$$

We examine the terms which appear in this sum. It is easy to see that $n_{e_{i_1\dots i_\ell}}(u_1,\dots,u_g)$ appears only once. We calculate the coefficient of $n_{e_{m_1\dots m_s}}(u_1,\dots,u_g)$ where $\{i_1,\dots,i_\ell\}\subsetneqq\{m_1,\dots,m_s\}$. The coefficient is

$$\sum_{k=\ell}^{s}\sharp\{\{j_1,\dots,j_k\}:\ \{i_1,\dots,i_\ell\}\subset\{j_1,\dots,j_k\}\subset\{m_1,\dots,m_s\}\}$$

and this is

$$\begin{aligned}\sum_{k=\ell}^{s}(-1)^{k-\ell}\binom{s}{k-\ell} &= \sum_{t=0}^{s-\ell}(-1)^t\binom{s-\ell}{t}\\ &= (1+(-1))^{s-\ell}\\ &= 0.\end{aligned}$$

Thus, the sum we are discussing turns out to consist of essentially the one term, that is, $n_{e_{i_1\dots i_\ell}}(u_1,\dots,u_g)$. This completes the proof of Lemma 6.

Now we describe the relations between the weight enumerator and the intersection enumerator as we claimed.

Theorem 7 *Let C be a code of length n. Then the following hold.*

(1)

$$W_C^{(g)}\left(x_{e_0}\leftarrow 1, x_{e_{j_1\dots j_k}}\leftarrow \prod_{\substack{\{i_1,\dots,i_\ell\}\subset\{j_1,\dots,j_k\}\\ 1\le j_1<\dots<j_k\le g}} X_{i_1\dots i_\ell}\ \textit{for}\ 1\le k\le g\right)$$
$$= I_C^{(g)}(\{X_{i_1 i_2\dots i_\ell}\}_{\substack{1\le\ell\le g\\ 1\le i_1<i_2<\dots<i_\ell\le g}}).$$

(2) $$x_{e_0}^n I_C^{(g)}\left(X_{j_1\cdots j_k}\leftarrow x_{e_0}^{(-1)^k}\prod_{\substack{\{i_1,\ldots,i_\ell\}\subset\{j_1,\ldots,j_k\}\\ 1\le i_1<\cdots<i_\ell\le g}} x_{e_{i_1}\cdots e_{i_\ell}}^{(-1)^{k-\ell}}\ \textit{for}\ 1\le k\le g\right)=W_C^{(g)}(\{x_a\}_{a\in\mathbf{F}_2^g}).$$

Proof. (1) We have that

$$\begin{aligned}
&W_C^{(g)}\left(x_{e_0}\leftarrow 1,\ x_{e_{j_1\cdots j_k}}\leftarrow \prod_{\substack{\{i_1,\ldots,i_\ell\}\subset\{j_1,\ldots,j_k\}\\ 1\le i_1<\cdots<i_\ell\le g}} X_{i_1\cdots i_\ell}\right)\\
&\quad=\sum_{u_1,\ldots,u_g\in C}\ \prod_{1\le j_1<\cdots<j_k\le g}\left(\prod_{\substack{\{i_1,\ldots,i_\ell\}\subset\{j_1,\ldots,j_k\}\\ 1\le i_1<\cdots<i_\ell\le g}} X_{i_1\cdots i_\ell}\right)^{e_{j_1\cdots j_k}(u_1,\ldots,u_g)}\\
&\quad=\sum_{u_1,\ldots,u_g\in C}\ \prod_{1\le j_1<\cdots<j_k\le g} X_{i_1\cdots i_\ell}^{\sum_{\substack{\{i_1,\ldots,i_\ell\}\subset\{j_1,\ldots,j_k\}\\ 1\le i_1<\cdots<i_\ell\le g}} n_{e_{j_1\cdots j_k}}}\\
&\quad=\sum_{u_1,\ldots,u_g\in C}\ \prod_{1\le j_1<\cdots<j_k\le g} X_{i_1\cdots i_\ell}^{wt(u_{i_1}\circ\cdots\circ u_{i_\ell})}\\
&\quad=I_C^{(g)}.
\end{aligned}$$

(2) First we observe a trivial relation

$$n_{e_0}(u_1,\ldots,u_g)=n-\sum_{1\le i_1<\cdots<i_\ell\le g} n_{e_{i_1\cdots i_\ell}}(u_1,\ldots,u_g).$$

Then we have that

$$
\begin{aligned}
W_C^{(g)} &= \sum_{u_1,\dots,u_g \in C} x_{e_0}^{n_{e_0}(u_1,\dots,u_g)} \prod_{1 \le i_1 < \cdots < i_\ell \le g} x_{e_{i_1 \dots i_\ell}}^{n_{e_1 \dots e_\ell}(u_1,\dots,u_g)} \\
&= \sum_{u_1,\dots,u_g \in C} x_{e_0}^{n - \sum_{1 \le i_1 < \cdots < i_\ell \le g} n_{e_{i_1 \dots i_\ell}}(u_1,\dots,u_g)} \prod_{1 \le i_1 < \cdots < i_\ell \le g} x_{e_{i_1 \dots i_\ell}}^{n_{e_1 \dots e_\ell}(u_1,\dots,u_g)} \\
&= x_{e_0}^n \sum_{u_1,\dots,u_g \in C} \prod_{1 \le i_1 < \cdots < i_\ell \le g} \left(\frac{x_{e_{i_1 \dots i_\ell}}}{x_{e_0}} \right)^{n_{e_{i_1 \dots i_\ell}}(u_1,\dots,u_g)} \\
&= x_{e_0}^n \sum_{u_1,\dots,u_g \in C} \prod_{1 \le i_1 < \cdots < i_\ell \le g} \left(\frac{x_{e_{i_1 \dots i_\ell}}}{x_{e_0}} \right)^{\sum_{\substack{\{i_1,\dots,i_\ell\} \subset \{j_1,\dots,j_k\} \\ 1 \le j_1 < \cdots < j_k \le g}} (-1)^{k-\ell} wt(u_{j_1} \circ \cdots \circ u_{j_k})} \\
&= x_{e_0}^n \sum_{u_1,\dots,u_g \in C} \prod_{1 \le j_1 < \cdots < j_k \le g} \left\{ \prod_{\substack{\{i_1,\dots,i_\ell\} \subset \{j_1,\dots,j_k\} \\ 1 \le i_1 < \cdots < i_\ell \le g}} \left(\frac{x_{e_{i_1 \dots i_\ell}}}{x_{e_0}} \right)^{(-1)^{k-\ell}} \right\}^{wt(u_{j_1} \circ \cdots \circ u_{j_k})} .
\end{aligned}
$$

Making use of

$$
\begin{aligned}
\prod_{\substack{\{i_1,\dots,i_\ell\} \subset \{j_1,\dots,j_k\} \\ 1 \le i_1 < \cdots < i_\ell \le g}} \left(\frac{1}{x_{e_0}} \right)^{(-1)^{k-\ell}} &= \left(\frac{1}{x_{e_0}} \right)^{(-1)^{k-1}\binom{k}{1} + (-1)^{k-2}\binom{k}{2} + \cdots + (-1)^{k-k}\binom{k}{k}} \\
&= \left(\frac{1}{x_{e_0}} \right)^{-(-1)^k} \\
&= x_{e_0}^{(-1)^k},
\end{aligned}
$$

we continue

$$
\begin{aligned}
& x_{e_0}^n \sum_{u_1,\dots,u_g \in C} \prod_{1 \le j_1 < \cdots < j_k \le g} \left(x_{e_0}^{(-1)^k} \prod_{\substack{\{i_1,\dots,i_\ell\} \subset \{j_1,\dots,j_k\} \\ 1 \le i_1 < \cdots < i_\ell \le g}} x_{e_{i_1 \dots i_\ell}}^{(-1)^{k-\ell}} \right)^{wt(u_{j_1} \circ \cdots \circ u_{j_k})} \\
& \quad = x_{e_0}^n I_C^{(g)} \left(X_{j_1 \dots j_k} \leftarrow x_{e_0}^{(-1)^k} \prod_{\substack{\{i_1,\dots,i_\ell\} \subset \{j_1,\dots,j_k\} \\ 1 \le i_1 < \cdots < i_\ell \le g}} x_{e_{i_1} \dots e_{i_\ell}}^{(-1)^{k-\ell}} \text{ for } 1 \le k \le g \right) .
\end{aligned}
$$

This completes the proof of Theorem 7.

We provide some examples.

$$W_C^{(2)}(x_{e_0} \leftarrow 1, x_{e_2} \leftarrow X_2, x_{e_1} \leftarrow X_1, x_{e_{12}} \leftarrow X_1X_2X_{12}) = I_C^{(2)}(X_1, X_2, X_{12}).$$

$$\begin{aligned} W_C^{(3)}(&x_{e_0} \leftarrow 1, x_{e_1} \leftarrow X_1, x_{e_2} \leftarrow X_2, x_{e_3} \leftarrow X_3, \\ &x_{e_{12}} \leftarrow X_1X_2X_{12}, x_{e_{13}} \leftarrow X_1X_3X_{13}, x_{e_{23}} \leftarrow X_2X_3X_{23}, x_{e_{123}} \leftarrow X_1X_2X_3X_{12}X_{13}X_{23}X_{123}) \\ &= I_C^{(3)}(X_1, X_2, X_3, X_{12}, X_{13}, X_{23}, X_{123}). \end{aligned}$$

$$x_{e_0}^n I_C^{(2)}(X_1 \leftarrow \frac{x_{e_1}}{x_{e_0}}, X_2 \leftarrow \frac{x_{e_2}}{x_{e_0}}, X_{12} \leftarrow \frac{x_{e_0}x_{e_{12}}}{x_{e_1}x_{e_2}}) = W_C^{(2)}(x_{e_0}, x_{e_1}, x_{e_2}, x_{e_{12}}).$$

$$\begin{aligned} x_{e_0}^n I_C^{(3)}(&X_1 \leftarrow \frac{x_{e_1}}{x_{e_0}}, X_2 \leftarrow \frac{x_{e_2}}{x_{e_0}}, X_3 \leftarrow \frac{x_{e_3}}{x_{e_0}}, X_{12} \leftarrow \frac{x_{e_0}x_{e_{12}}}{x_{e_1}x_{e_2}}, X_{13} \leftarrow \frac{x_{e_0}x_{e_{13}}}{x_{e_1}x_{e_3}}, X_{23} \leftarrow \frac{x_{e_0}x_{e_{23}}}{x_{e_2}x_{e_3}}, \\ &X_{123} \leftarrow \frac{x_{e_1}x_{e_2}x_{e_3}x_{e_{123}}}{x_{e_0}x_{e_{12}}x_{e_{13}}x_{e_{23}}}) \\ &= W_C^{(3)}(x_{e_0}, x_{e_1}, x_{e_2}, x_{e_{12}}, x_{e_3}, x_{e_{13}}, x_{e_{23}}, x_{e_{123}}). \end{aligned}$$

It is known that Jacobi forms appear in the Fourier-Jacobi expansion of Siegel modular forms (*cf.* [5, 15]). Actually this is one of Ozeki's motivations for his Jacobi polynomial. In order to get a similar result for the g-th Jacobi polynomial, we consider the $(g+1)$-th intersection enumerator. For a code of length n, it holds

$$\begin{aligned} I_C^{(g+1)} &= \sum_{u_1,\dots,u_{g+1}\in C} X_1^{wt(u_1)} \dots X_g^{wt(u_g)} X_{g+1}^{wt(u_{g+1})} \prod_{2\le\ell\le g+1} \left(\prod_{1\le k_1<\cdots<k_\ell\le g+1} X_{k_1\dots k_\ell}^{wt(u_{k_1}\circ\cdots\circ u_{k_\ell})} \right) \\ &= \sum_{u_{g+1}\in C} \left\{ \sum_{u_1,\dots,u_g\in C} \prod_{2\le\ell\le g+1} \left(\prod_{1\le k_1<\cdots<k_\ell\le g+1} X_{k_1\dots k_\ell}^{wt(u_{k_1}\circ\cdots\circ u_{k_\ell})} \right) \right\} X^{wt(u_{g+1})} \\ &= \sum_{u_{g+1}\in C} Jac^{(g)}(C, u_{g+1}) X_{g+1}^{wt(u_{g+1})} \\ &= \sum_{r=0}^{n} \left(\sum_{\substack{v\in C \\ wt(v)=r}} Jac^{(g)}(C, v) \right) X_{g+1}^r. \end{aligned}$$

We have thus obtained the following

Theorem 8 *Let C be a code of length n. Putting $X_{g+1} = Y$, we have*

$$I_C^{(g+1)} = \sum_{r=0}^{n} \left(\sum_{\substack{v \in C \\ wt(v)=r}} Jac^{(g)}(C, v) \right) Y^r.$$

In this theorem, the g-th Jacobi polynomial is obtained from the $(g+1)$-th weight enumerator. If we ask any relation between those of the same genus, we get the following

Proposition 9 *The g-th Jacobi polynomial of a code C with the zero reference vector is the g-th intersection enumerator, that is,*

$$Jac^{(g)}(C, 0) = I_C^{(g)}.$$

Proof. Since we have $wt(u_{k_1} \circ \cdots \circ u_{k_\ell}) = 0$ for $k_\ell = g+1$ from the assumption, it holds

$$\prod_{2 \leq \ell \leq g+1} \left(\prod_{\substack{1 \leq k_1 < \cdots < k_\ell \leq g+1 \\ u_{g+1}=v}} X_{k_1 \ldots k_\ell}^{wt(u_{k_1} \circ \cdots \circ u_{k_\ell})} \right) = \prod_{2 \leq \ell \leq g} \left(\prod_{1 \leq k_1 < \cdots < k_\ell \leq g} X_{k_1 \ldots k_\ell}^{wt(u_{k_1} \circ \cdots \circ u_{k_\ell})} \right).$$

Therefore we have

$$\begin{aligned} Jac^{(g)}(C, 0) &= \sum_{u_1, \ldots, u_g \in C} \left(\prod_{1 \leq i \leq g} X_i^{wt(u_i)} \right) \left\{ \prod_{2 \leq \ell \leq g+1} \left(\prod_{\substack{1 \leq k_1 < \cdots < k_\ell \leq g+1 \\ u_{g+1}=v}} X_{k_1 \ldots k_\ell}^{wt(u_{k_1} \circ \cdots \circ u_{k_\ell})} \right) \right\} \\ &= \sum_{u_1, \ldots, u_g \in C} \left(\prod_{1 \leq i \leq g} X_i^{wt(u_i)} \right) \left\{ \prod_{2 \leq \ell \leq g} \left(\prod_{1 \leq k_1 < \cdots < k_\ell \leq g} X_{k_1 \ldots k_\ell}^{wt(u_{k_1} \circ \cdots \circ u_{k_\ell})} \right) \right\} \\ &= I_C^{(g)}. \end{aligned}$$

This completes the proof Proposition 9.

Proposition 10 *For $i = 1, 2$, let C_i be a code of length n_i and v_i an element of $\mathbf{F}_2^{n_i}$. Then we have*

$$Jac^{(g)}(C_1, v_1) Jac^{(g)}(C_2, v_2) = Jac^{(g)}(C_1 \oplus C_2, v_1 \oplus v_2).$$

Proof. Straightforward.

We conclude this paper with some explicit calculations. Let H be a code of length 8 defined by the generator matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

This is self-dual and doubly even. For simplicity, we put

$$a = X_1, b = X_2, c = X_{12}, d = X_{13}, e = X_{23}, f = X_{123}.$$

As the reference vectors we take

$$v_1 = (0,0,0,0,0,0,0,0),\ v_2 = (1,1,1,1,0,0,0,0),\ v_3 = (1,0,0,0,0,0,0,0).$$

Terms are grouped following the calculation made after Proposition 4.

$$Jac^{(2)}(H, v_1) = I_H^{(2)} = (a^8b^8c^8+a^8+b^8+1)+14(a^8b^4c^4+b^4)+14(a^4b^8c^4+a^4)+14(a^4b^4c^4+a^4b^4)+168a^4b^4c^2.$$

$$\begin{aligned} Jac^{(2)}(H, v_2) = {} & (a^8b^8c^8d^4e^4f^4+b^8e^4+a^8d^4+1)+(a^8b^4c^4d^4e^4f^4+b^4e^4+a^8b^4c^4d^4+b^4) \\ & + 12(a^8b^4c^4d^4e^2f^2 + b^4e^2) + (a^4b^8c^4d^4e^4f^4 + a^4b^8c^4e^4 + a^4d^4 + a^4) \\ & + 12(a^4b^8c^4d^2e^4f^2 + a^4d^2) + (a^4b^4c^4d^4e^4f^4 + a^4b^4e^4 + a^4b^4d^4 + a^4b^4c^4) \\ & + 12(a^4b^4c^4d^2e^2f^2 + a^4b^4d^2e^2) + 12(a^4b^4c^2d^4e^2f^2 + a^4b^4c^2e^2) \\ & +12(a^4b^4c^2d^2e^4f^2+a^4b^4c^2d^2)+12(a^4b^4c^2d^2e^2f^2+a^4b^4c^2d^2e^2)+96a^4b^4c^2d^2e^2f. \end{aligned}$$

$$\begin{aligned} Jac^{(2)}(H, v_3) = {} & (a^8b^8c^8def+b^8e+a^8d+1)+7(a^8b^4c^4def+b^4e+a^8b^4c^4d+b^4) \\ & + 7(a^4b^8c^4def + a^4b^8c^4e + a^4d + a^4) + 7(a^4b^4c^4def + a^4b^4e + a^4b^4d + a^4b^4c^4) \\ & + 42(a^4b^4c^2def + a^4b^4c^2e + a^4b^4c^2d + a^4b^4c^2). \end{aligned}$$

Acknowledgment. This work was supported by JSPS KAKENHI Grant Number 17K05164. The computations were done with Magma [2] and Maple.

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Canon Imaging Systems Inc.
Japan
keita.homma12f2@gmail.com

Institute of Science and Engineering
Kanazawa University
Japan
oura@se.kanazawa-u.ac.jp