Invariant theoretical approaches to the ring of Siegel modular forms and the related topics(a survey)

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This is a fairly precise reproduction of my talk at the conference. My intention is to present some of our results without going into details, gathering the related ones. We are concerned with the theory of modular forms, coding theory, invariant theory and theta series.

1. Preliminaries. The notations are standard. Let $\mathfrak{S}_g$ denote the Siegel upper-half space and $\Gamma_g = Sp_{2g}(\mathbf{Z})$ the Siegel modular group. In the following, we shall express a typical element M of $Sp_{2g}(\mathbf{R})$ in the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

by four $g \times g$ matrices $a,\ b,\ c,\ d$. If n is a positive integer, a subgroup $\Gamma_g(n)$ of Γ_g defined by the condition $M \equiv 1 \pmod{n}$ is called the principal congruence group of level n. For an even positive integer n, we shall define a subgroup $\Gamma_g(n, 2n)$ of $\Gamma_g(n)$ by the condition $(b)_0 \equiv (d)_0 \equiv 0 \pmod{2l}$, in which $(\ \cdot\ )_0$ denotes a vector of its diagonal coefficients.

We consider a discrete subgroup Γ of $Sp_{2g}(\mathbf{R})$ which is "commensurable" with Γ_g. A holomorphic function ψ on $\mathfrak{S}_g$ satisfying

$$\psi((a\tau + b)(c\tau + d)^{-1}) = \det(c\tau + d)^k \psi(\tau)$$

for every element M in Γ is called a modular form of weight k for Γ. In this definition, we have to assume that ψ is holomorphic at cusps in the case $g = 1$. The set of such functions forms a vector space $A(\Gamma)_k$ over $\mathbf{C}$ and the graded ring of modular forms of integral weights for Γ is denoted by

$$A(\Gamma) = \bigoplus_{0 \le k < \infty} A(\Gamma)_k,\ A(\Gamma)_0 = \mathbf{C}.$$

For a graded ring $S = S_0 \oplus S_1 \oplus \cdots$, we put $S^{(d)} = S_0 \oplus S_d \oplus S_{2d} \oplus \cdots$.

We shall write vectors in $\mathbf{Q}^{2g}$ by m and use $m',\ m''$ to denote the first and second g coefficients in m. If τ is a point of $\mathfrak{S}_g$, the function

$$\theta_m(\tau) = \sum_{p \in \mathbf{Z}^g} \exp 2\pi\sqrt{-1} \left\{ \frac{1}{2} {}^t(p + \frac{m'}{2})\tau(p + \frac{m'}{2}) + {}^t(p + \frac{m'}{2})\frac{m''}{2} \right\}$$

is called the theta-constants of characteristic m.

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2. History on $A(\Gamma_g)$, $g = 2, 3$. First we collect some papers.

['**39 Siegel**] Siegel, C.L., Einführung in die Theorie der Modulfunktionen n-ten Grades, Math. Ann. 116, 617-657 (1939).

['**56 Satake**] Satake, I., On the compactification of the Siegel space, J. Indian Math. Soc., n. Ser. 20, 259-281 (1956).

['**58 Baily**] Baily, W.L., Satake's compactification of V_n, Am. J. Math. 80, 348-364 (1958).

['**57-'58 Cartan**] Séminaire de Cartan, Fonctions automorphes, E.N.S. (1957-1958).

['**60 Igusa**] Igusa, J., Arithmetic variety of moduli for genus two, Ann. Math. (2) 72, 612-649 (1960).

['**62 Igusa**] Igusa, J., On Siegel modular forms of genus two, Am. J. Math. 84, 175-200 (1962). The ring $A(\Gamma_2)^{(2)}$ is determined.

['**64-1 Igusa**] Igusa, J., On the graded ring of theta-constants, Am. J. Math. 86, 219-246 (1964). A "fundamental lemma" is given.

['**64-2 Igusa**] Igusa, J., On Siged modular forms of genus two II, Am. J. Math. 86, 392-412 (1964). The fundamental lemma in ['64-1 Igusa] is used to determine $A(\Gamma_2)$.

['**66 Igusa**] Igusa, J., On the graded ring of theta-constants II, Am. J. Math. 88, 221-236 (1966). The fundamental lemma in ['64-1 Igusa] is generalized.

['**67 Igusa**] Igusa J., Modular forms and projective invariants, Am. J. Math. 89, 817-855 (1967). The ρ-homomorphism is introduced. The ring structures $A(\Gamma_g)$ for $g = 1, 2$ are reproved.

['**86 Tsuyumine**] Tsuyumine, S., On Siegel modular forms of degree three, Am. J. Math. 108, 755-862; Addendum 1001-1003 (1986). Using the ρ-homomorphism, he determined the dimension formula and the generators of $A(\Gamma_3)$. The relations among the generators are not given.

In the paper ['62 Igusa], he says, after describing the importance of $A(\Gamma_g)$, "..., we knew very little about this ring beyond the facts that it is finitely generated and that an operator Φ introduced by Siegel is almost an epimorphism of the graded rings of degree n to degree $n-1$." At the time of his writing ['62 Igusa], it is known

$$\dim A(\Gamma_2)_2 = 0, \dim A(\Gamma_2)_4 = \dim A(\Gamma_2)_6 = \dim A(\Gamma_2)_8 = 1,$$

which are proved by Maass [22] and partly by Witt [39]. Applying the moduli theory ['60 Igusa] of curves in genus two, he obtains that $A(\Gamma_2)^{(2)}$ can be generated by Eisenstein series of weights four, six, ten and twelve. These four Eisenstein series are algebraically independent over $\mathbf{C}$ and the dimension formula of $A(\Gamma_2)^{(2)}$ also follows. In ['64-1 Igusa], he showed that the graded ring $A(\Gamma_g(4,8))$ is the normalization of the graded ring $\mathbf{C}[\theta_m\theta_n]$ generated over $\mathbf{C}$ by the products of theta-constants with $m, n \in \mathbf{Z}^{2g}$ i.e.,

$$A(\Gamma_g(4,8)) = (\mathbf{C}[\theta_m\theta_n])^N,$$

in which N denotes the integral closure of the ring in its field of fractions. This is called by him a fundamental lemma. In ['64-2 Igusa], it is shown that the ring $\mathbf{C}[\theta_m\theta_n]$ is normal in $g=2$, which coincides with $A(\Gamma_2(4,8))$ by the fundamental lemma. Finally Going-down process, i.e., taking successively the invariant subrings of finite groups, provides $A(\Gamma_2)$. As a consequence, we know that the ring $A(\Gamma_2)$ can be generated by five elements and that there is essentially a unique relation among the generators. The explicit formula of this relation can be found in ['67 Igusa].

The fundamental lemma is generalized in ['66 Igusa] as

$$A(\Gamma_g(r^2, 2r^2)) = (\mathbf{C}[\theta_m\theta_n])^N$$

with $\frac{rm}{2}, \frac{rn}{2} \in \mathbf{Z}^{2g}$ for any even positive integer r. This generalization, again with Going-down process, enables us to investigate $A(\Gamma_g(l))$ for any l.

In ['67 Igusa], he defined the ρ-homomorphism from a subring of $A(\Gamma_g)$ to $S(2, 2g+2)$. We give a quick view on $S(2, 2g+2)$ later. This ρ-homomorphism gives a bijection from $A(\Gamma_1)$ to $S(2,4)$, an injection from $A(\Gamma_2)$ to $S(2,6)$. In this way, he obtains the structure theorems of $A(\Gamma_g)$, $g = 1, 2$. In $g=3$, he determines the kernel of ρ, which is a principal ideal, and the vector spaces of low weights. The latter leads to the affirmative answer of a problem of Witt [39], which asks whether two theta series of even unimodular lattices E_8^2 and D_{16}^+ in genus 3 coincide, i.e., whether we have $\vartheta_{E_8^2} = \vartheta_{D_{16}^+}$ in genus 3. This problem is independently settled by Kneser

[21]. See also [5]. The problem of Witt in the case of rank 24 is considered in [12], [5]. Their theorem says that theta series of twenty-four even unimodular lattices of rank 24 are linearly independent if and only if $g \geq 12$. We will come back to this problem in the next section.

We conclude this section by describing the graded ring $S(2,r)$ of projective invariants of a binary form

$$f = u_0 x^r + u_1 x^{r-1} y + u_2 x^{r-2} y^2 + \cdots + u_r y^r.$$

We refer to [36] in more details. Putting $f(u, \sigma x) = f(\tilde{u}, x)$ for $\sigma \in SL_2(\mathbf{C})$, we have an irreducible representation of $SL_2(\mathbf{C})$ of degree $r+1$. Under this representation, we get the invariant ring $S(2,r)$. For example, we consider $f = u_0 x^2 + u_1 xy + u_2 y^2$. For

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$\begin{aligned}
\sigma f &= u_0(ax+by)^2 + u_1(ax+by)(cx+dy) + u_2(cx+dy)^2 \\
&= (u_0 a^2 + u_1 ac + u_2 c^2)x^2 + (2u_0 ab + u_1(ad+bc) + 2u_2 cd)xy \\
&\quad + (u_0 b^2 + u_1 bd + u_2 d^2)y^2 \\
&= \tilde{u_0} x^2 + \tilde{u_1} xy + \tilde{u_2} y^2,
\end{aligned}$$

i.e.,

$$\begin{pmatrix} \tilde{u_0} \\ \tilde{u_1} \\ \tilde{u_2} \end{pmatrix} = \begin{pmatrix} a^2 & ac & c^2 \\ 2ab & ad+bc & 2cd \\ b^2 & bd & d^2 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}.$$

The ring $S(2,4)$ is a set of polynomials $f(u_0, u_1, u_2)$ such that $f(\tilde{u_0}, \tilde{u_1}, \tilde{u_2}) = f(u_0, u_1, u_2)$ for any

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $SL_2(\mathbf{C})$. Investigation (finite generation, ring structures, etc.) of $S(n,r)$ is one of the main topics in the 19th century. For example,

$$S(2,6) = \mathbb{C}[\underbrace{A_2, B_4, C_6, D_{10}}_{\text{alg. indep.}}, E_{15}]/\left(E^2 - P(A,B,C,D) - Q(A,B,C,D)E\right),$$

in which P and Q are certain polynomials. This ring plays an important role in ['60 Igusa], ['62 Igusa], ['67 Igusa] The ring $S(2,8)$ is determined by Shioda [37] as follows:

$$S(2,8) = \mathbb{C}[\underbrace{J_2, J_3, J_4, J_5, J_6, J_7}_{\text{alg. indep.}}, J_8, J_9, J_{10}]/\{5 \ \text{ relations}\}.$$

This ring is used to determine $A(\Gamma_3)$ in ['86 Tsuyumine]. In the above two examples, the subscripts denote the degrees of homogeneous polynomials.

3. Coding theory, theta relations, and some graded rings. First we recall the theory of binary codes. We shall denote by $\mathbf{F}_2$ the field of two elements 0, 1. A subspace of $\mathbf{F}_2^n$ is called a linear code of length n. In the rest of this note, we omit "linear". A code C is called self-dual if it coincides with its dual $C^\perp = \{x \in \mathbf{F}_2^n | \ x \cdot y = \sum x_i y_i = 0, \ \forall y \in C\}$. A weight $wt(x)$ of $x \in \mathbf{F}_2^n$ is the number of the non-zero coordinates of x. A code C is called doubly-even if the weight of every element in C is a multiple of 4. If a code is self-dual and doubly-even, it is said to be Type II. It is known that Type II codes exist if and only if the length n is a multiple of 8. Two codes is said to be equivalent if one is obtained from another under some coordinate change. Classifications of non-equivalent Type II codes are known up to $n = 32$ (Pless, Conway, Sloane, see [8] and its references).

$n = 8$	unique e_8
$n = 16$	2 classes e_8^2, d_{16}^+
$n = 24$	9 classes
$n = 32$	85 classes
$n \geq 40$	open

We give the first example of Type II code. That is a vector space e_8 of length 8 generated by four elements

$$\begin{gathered}(1,1,1,1,0,0,0,0),\\(0,0,1,1,1,1,0,0),\\(0,0,0,0,1,1,1,1),\\(1,0,1,0,1,0,1,0).\end{gathered}$$

The weight enumerator of C of length n is defined by

$$W_C(x, y) = \sum_{v \in C} x^{n - wt(v)} y^{wt(v)}.$$

For example, the weight enumerator of e_8 is

$$W_{e_8}(x, y) = x^8 + 14x^4y^4 + y^8.$$

The weight enumerator of a code C and that of its dual $C^\perp$ are related as

$$W_{C^\perp}(x, y) = \frac{1}{|C|} W_C(x + y, x - y),$$

in which $|C|$ denotes the cardinality of C. This is called the MacWilliams identity.

Next we shall consider the graded ring over $\mathbf{C}$ generated by the weight enumerators of Type II codes. This ring coincides with the invariant ring of the finite group generated by

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}.$$

This is from [17]. This group is a finite unitary reflection group of order 192, known as No.9 in [38]. The map

$$x \longmapsto \theta_{00}(2\tau),\ y \longmapsto \theta_{10}(2\tau)$$

induces the isomorphism between this invariant ring and $A(\Gamma_1)^{(4)}$, see [6]. In particular, the weight enumerator $W_C(x,y)$ of any Type II code C of length n gives a modular form of weight $n/2$ for Γ_1. If we consider an index 2 subgroup of No.9, known as No.8 in [38], the invariant ring can not be generated by the weight enumerators, however, the above map gives the isomorphism between the invariant ring of this subgroup and the ring $A(\Gamma_1) = A(\Gamma_1)^{(2)}$.

We introduce the concept of genus in the weight numerators. What we have stated above corresponds to the case genus one. The weight enumerator of a code C in genus g is defined by

$$W_C^{(g)}(x_a : a \in \mathbf{F}_2^g) = \sum_{x_1,\cdots,x_g \in C} \prod_{a \in \mathbf{F}_2^g} x_a^{n_a(x_1,\cdots,x_g)},$$

where

$$n_a(x_1,\cdots,x_g) = \sharp\{i | a = (x_{1i}, x_{2i}, \cdots, x_{gi})\}.$$

Here, as we promised, we consider the problem of Witt in coding theory. The general method to attack this problem is given by Nebe [23]. See also [32], [10], [26]. There exist 1, 2, 9, 85 classes of Type II codes of length 8, 16, 24, 32, respectively. For length 16, two weight enumerators are linearly independent if and only if $g \geq 3$. For length 24, nine weight enumerators are linearly independent if and only if $g \geq 6$. For length 32, eighty-five weight enumerators are linearly independent if and only if $g \geq 10$.

The Gleason type theorem for $W_C^{(g)}$ also holds, i.e., the ring over $\mathbf{C}$ generated by the weight enumerators $W_C^{(g)}$ of all Type II codes coincides with the invariant ring of the finite group G_g for any positive integer g. If we consider a index 2 subgroup H_g of G_g, the invariant ring of H_g is not generated by the weight enumerators any more, but Runge [32] showed

$$A(\Gamma_g)^{(2)} \cong \left(\mathbf{C}[x_a]^{H_g}/\{\text{theta relations}\}\right)^N.$$

We omit the precise definitions of these groups. For a Type II code C of length n, $W_C^{(g)}(\theta_{a0}(2\tau))$ is a modular form of weight $n/2$ for Γ_g.

Runge([32]. *cf.* [15]) obtained the following theorem:

$$A(\Gamma_3) \cong \mathbf{C}[x_a]^{H_3}/(W^{(3)}_{d_{16}^+} - W^{(3)}_{e_8^2}).$$

From this, the dimension formula of $A(\Gamma_3)$ reduces to the computation of the dimension formula of the invariant ring of H_3, i.e.,

$$\sum_{d\geq 0} (\dim_{\mathbf{C}} A(\Gamma_3)_d)\, t^{2d} = \left\{ \sum_{d'\geq 0} \left(\dim_{\mathbf{C}}(\mathbf{C}[x_a]^{H_3})_{d'}\right) t^{d'} \right\} \times (1 - t^{16}).$$

Note that an exponent of t in the left-hand side is $2d$. In [33], the dimension formula of the invariant ring of H_3 is computed and the dimension formula of $A(\Gamma_3)$ in ['86 Tsuyumine] is reproved.

If we view the works of Igusa and of Runge, we can find the "normalization type" theorems in both. However, Igusa starts from $\theta_m(\tau)$, while Runge from $\theta_{m'0}(2\tau)$.

Next we shall give some results in genus 4. The degree 24 part of the invariant ring of H_4 is spanned by the weight enumerators of Type II codes of length 24. There are 9 Type II codes of this length and the dimension of the vector space over $\mathbf{C}$ spanned by the weight enumerators of those codes is 7, see [25]. On the other hand, the dimension of $A(\Gamma_4)_{12}$ is 6, see [28]. In [16], we analysed the map

$$(\mathbf{C}[x_a]^{H_4})_{24} = \langle W^{(4)}_{C_1}, W^{(4)}_{C_2}, \cdots, W^{(4)}_{C_9}\rangle_{\mathbf{C}} \longrightarrow A(\Gamma_4)_{12}$$

given by $x_a \mapsto \theta_{a0}(2\tau)$. This linear map is surjective and the kernel is explicitly described. The obtained relation gives a non-trivial cusp form in genus 5. It is an open problem if this relation comes from Riemann's theta relations.

The same problem in the case of length 32 is treated in [26]. The dimension of the degree 32 part of the invariant ring $\mathbf{C}[x_a]^{H_4}$ is 19, while the dimension of the corresponding vector space of modular forms is 14. See [25], [30], respectively. Finally we get 5 relations among theta series of weight 16 in genus 4. In the course of these computations, we use the Restriction technique ([29], [30]) to overcome a difficulty which does not appear in [16].

The following picture is helpful for understanding what we have said.

$$\begin{array}{ccc} \text{code} & \xrightarrow{\text{“Construction A”}} & \text{lattice} \\ {\scriptstyle C\mapsto W_C^{(g)}}\downarrow & & \downarrow{\scriptstyle \text{theta}} \\ \text{invariant polynomial} & \xrightarrow{x_a\mapsto\theta_{a0}(2\tau)} & \text{modular form} \end{array}$$

For this diagram, see Conway-Sloane [9], Runge [34], Ebeling [11], Bannai-Ozeki [2], Bannai-Dougherty-Harada-Oura [3], Bannai-Harada-Ibukiyama-Munemasa-Oura [1], Betsumiya-Choie [4], etc. For Gleason type theorem, see Runge [34], Rains-Sloane [31] Nebe-Rains-Sloane [24], Chiera [7], etc.

As far as we concerned with even unimodular lattices, $A(\Gamma_g)^{(4)}$ might be suitable. This ring is the normalization of the ring generated over $\mathbf{C}$ by theta series of even unimodular lattices, i.e., $A(\Gamma_g)^{(4)} = \mathbf{C}[\vartheta_\wedge]^N$, see [14]. No normalization is necessary if $g \leq 4$, see [35]. Explicit structures are known as follows.

$$A(\Gamma_1)^{(4)} = \mathbf{C}[E_4, \Delta] \text{ by Hecke [18]},$$
$$A(\Gamma_2)^{(4)} = \mathbf{C}[\vartheta_{E_8}, \vartheta_{A_1^{24}}, \vartheta_{D_{24}^+}, \vartheta_{D_{32}^+}, \vartheta_{D_{40}^+}] \text{ by Ozeki [27].}$$

Next we consider the ring $A_{\mathbf{Z}}(\Gamma_g)$ over $\mathbf{Z}$ generated by the modular forms whose Fourier coefficients are all integral. It is known that this ring is finitely generated over $\mathbf{Z}$, see [13]. Explicit structures are known as follows.

$$A_{\mathbf{Z}}(\Gamma_1) = \mathbf{Z}[E_4, E_6, \Delta],$$
$$A_{\mathbf{Z}}(\Gamma_2) = \mathbf{Z}[\underbrace{X_4, \cdots, X_{48}}_{\text{15 generators}}] \text{ by Igusa [20].}$$

We conclude this note by describing a result with Prof.Choie concerning to the weight enumerators in genus 2. Actually this investigation comes from a question raised by Prof.Skoruppa when I visited Bordeaux in December 1998. It is known that the ring over $\mathbf{C}$ by the weight enumerators of Type II codes is generated by five elements $W_{e_8}^{(2)}, W_{g_{24}}^{(2)}, W_{d_{24}^+}^{(2)}, W_{d_{32}^+}^{(2)}, W_{d_{40}^+}^{(2)}$. Therefore the weight enumerator of any Type II code C can be expressed by these elements. Now, we are interested in the coefficients of polynomial expressions of the weight enumerators by these five elements. The result could be stated as follows. Let R be a subring of $\mathbf{C}$. Then the equality

$$R[W_C^{(2)}] = R[W_{e_8}^{(2)}, W_{g_{24}}^{(2)}, W_{d_{24}^+}^{(2)}, W_{d_{32}^+}^{(2)}, W_{d_{40}^+}^{(2)}]$$

holds if and only if

$$\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{43}] \subseteq R.$$

In the proof of this theorem, the above identity on $A_{\mathbf{Z}}(\Gamma_2)$ is applied.

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