The Tutte polynomials of genus g

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Abstract

In the paper [Proceedings of the Japan Academy, Ser. A Mathematical Sciences, 95(10) 111-113], the authors introduce the concept of the Tutte polynomials of genus g and announce that each matroid M can be reconstructed from its Tutte polynomial of genus $|\mathcal{B}(M)|$, where $\mathcal{B}(M)$ denotes the family of bases of M. In that paper, we also announced that, for all g, there exist inequivalent matroids that have the same Tutte polynomial of genus g. In this paper, we prove these theorems.

Key Words and Phrases. matroid, Tutte polynomial.

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1 Introduction

Let E be a finite set. A *matroid* M on $E = E_M$ is a pair $(E, \mathcal{I})$, where $\mathcal{I}$ is a non-empty family of subsets of E with the following three conditions:

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(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$;

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

For a matroid M, the set E_M is called a *ground set* of M. The family $\mathcal{I}$ is called the *independent sets.* A matroid $(E, \mathcal{I})$ is *isomorphic* to another matroid $(E', \mathcal{I}')$ if there is a bijection φ from E to E' such that $\varphi(I) \in \mathcal{I}'$ holds for each member $I \in \mathcal{I}$, and $\varphi^{-1}(I') \in \mathcal{I}$ holds for each member $I' \in \mathcal{I}'$.

It follows from the condition (I3) that all maximal independent sets in a matroid M have the same cardinality k, called the *rank* of M. The maximal independent sets $\mathcal{B}(M)$ are called the *bases* of M. The *rank* $\rho(A)$ of an arbitrary subset A of E is the cardinality of the largest independent set contained in A. A *circuit* in a matroid M is a minimal dependent subset of E. A *loop* in a matroid M is a singleton $\{e\} \subset E$ such that $\rho(\{e\}) = 0$, that is, a circuit of size 1. Let us denote the family of circuits of M by $\mathcal{C}(M)$. There are many other known axiomatic systems of matroids, all of which are equivalent. The following axiomatic system for circuits is also frequently used in this paper.

(C1) $\emptyset \notin \mathcal{C}(M)$.

(C2) If C_1 and C_2 are members of $\mathcal{C}(M)$ and $C_1 \subseteq C_2$, then $C_1 = C_2$;

(C3) If $C_1, C_2 \in \mathcal{C}(M)$, $C_1 \neq C_2$, and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}(M)$ such that $C_3 \subseteq (C_1 \cap C_2) \setminus \{e\}$.

The above condition (C3) is called the *circuit elimination axiom.*

For a matroid M, there exists a unique matroid M^* such that $E_{M^*} = E_M$, and that a set $S \subset E_{M^*}$ is independent if and only if there exists a basis $B \in \mathcal{B}(M)$ which satisfies $S \cap B = \emptyset$. Such matroid M^* is called the *dual matroid* of M. The bases of M^* is defined by $\{E_M \setminus B | B \in \mathcal{B}(M)\}$, which is called the *cobases* of M. It is known that $(M^*)^* = M$. A *cocircuit* and a *coloop* of a matroid M mean a circuit and a loop of M^*, respectively. The rank of M^* is called the *corank* of M.

A matroid is *self-dual* if M^* is isomorphic to M.

We give examples.

Example 1.1. Let A be a $k \times n$ matrix over a field. This gives a matroid M on the set $E = \{1, \ldots, n\}$, in which a set ℓ is independent if and only if the family of columns of A whose indices belong to ℓ is linearly independent. Such a matroid is called a vector matroid.

Example 1.2. Let $E = \{1, \ldots, n\}$ and r a non-negative integer. We define a matroid on E by taking every r-element subset of E to be a base. This is known as the uniform matroid $U_{r,n}$.

A matroid is *empty* if it is isomorphic to $U_{0,0}$; otherwise it is *non-empty.* A matroid $M = (E, \mathcal{I})$ is *separable* if there exists two non-empty matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ with E_1 and E_2 disjoint such that $E := E_1 \dot\cup E_2$ and $\mathcal{I} := \{I_1 \dot\cup I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$ hold. In this case, let us denote $M = M_1 \dot\cup M_2$. A matroid is *non-separable* if it is not separable. It is known that any matroid can be uniquely decomposed into non-separable matroids, which are called the *components* of the matroid.

A matroid $M = \dot\cup M_i$ is *equivalent* to another matroid $M' = \dot\cup M'_i$ if, with reordering if necessary, each M'_i is isomorphic to either M_i or M_i^*.

Let M be a matroid on the set E, having rank function ρ. The Tutte polynomial of M is defined as follows:

$$T(M; x, y) = \sum_{A \subset E} (x-1)^{\rho(E) - \rho(A)} (y-1)^{|A| - \rho(A)}.$$

By definition of the dual matroid, $T(M^*; x, y) = T(M; y, x)$ holds.

For example, the Tutte polynomial of the uniform matroid $U_{r,n}$ is

$$T(U_{r,n}; x, y) = \sum_{i=0}^{r} \binom{n}{i} (x-1)^{r-i} + \sum_{i=r+1}^{n} \binom{n}{i} (y-1)^{i-r}.$$

It is well known that, from the Tutte polynomial of a matroid M, we can read out all of the rank, the corank, the number of loops, the number of the coloops, the number of bases (cobases) of M ([14]).

It is easy to show that $T(M; x, y)$ is a matroid invariant and if two matroids have the same Tutte polynomial, we call them *T-equivalent.* It is also known that there exist two non-isomorphic matroids, which are T-equivalent ([14]).

This gives rise to a natural question: is there a generalization of the Tutte polynomial? We now present the concept of the Tutte polynomial of genus g. (This definition already appeared in the announcement paper [11].)

Definition 1.3. Let $M := (E, \mathcal{I})$ be a matroid. Let $(A_i \mid i = 1, \ldots, g)$ be an arbitrary g-tuple of subsets of E. Let $\Lambda_1 := \{1, \ldots, g\}$ and let $\Lambda_2 := \binom{\Lambda_1}{2}$. For every element $\lambda \in \Lambda_2$, let us denote $A_{\cap(\lambda)} := \cap_{i\in\lambda} A_i$ and $A_{\cup(\lambda)} := \cup_{i\in\lambda} A_i$. By using these notations, let us define

$$
\begin{aligned}
T^{(g)}(A_1, \ldots, A_g) \quad := \quad & \prod_{\lambda \in \Lambda_1} (x_\lambda - 1)^{\rho(E)-\rho(A_\lambda)} (y_\lambda - 1)^{|A_\lambda| - \rho(A_\lambda)} \\
& \times \prod_{\lambda \in \Lambda_2} (x_{\cap(\lambda)} - 1)^{\rho(E)-\rho(A_{\cap(\lambda)})} (y_{\cap(\lambda)} - 1)^{|A_{\cap(\lambda)}| - \rho(A_{\cap(\lambda)})} \\
& \times \prod_{\lambda \in \Lambda_2} (x_{\cup(\lambda)} - 1)^{\rho(E)-\rho(A_{\cup(\lambda)})} (y_{\cup(\lambda)} - 1)^{|A_{\cup(\lambda)}| - \rho(A_{\cup(\lambda)})}
\end{aligned}
$$

Then the genus g Tutte polynomial $T^{(g)}(M)$ of the matroid M will be defined as follows:

$$
\begin{aligned}
T^{(g)}(M) &:= T^{(g)}(M; x_{\lambda_1}, y_{\lambda_1}, x_{\cap\lambda_2}, y_{\cap\lambda_2}, x_{\cup\lambda_2}, y_{\cup\lambda_2} : \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2) \qquad (1.1) \\
&:= \sum_{A_1, \ldots, A_n \subseteq E} T^{(g)}(A_1, \ldots, A_g)
\end{aligned}
$$

For any positive integer g, we have

$$
T^{(g)}(M) = \frac{1}{2^{|E_M|}} T^{(g+1)}(M)|_{x_{A_{g+1}}=2, y_{A_{g+1}}=2, x_{A_j \cap A_{g+1}}=2 (j=1,\ldots,g), y_{A_k \cup A_{g+1}}=2 (k=1,\ldots,g)}.
$$

It is straightforward to see that $T^{(g)}$ is a matroid invariant. For example, the genus 2 Tutte polynomial $T^{(2)}(M)$ of the matroid M is as follows:

$$
\begin{aligned}
& T(M; x_1, x_2, x_{\cap\{1,2\}}, x_{\cup\{1,2\}}, y_1, y_2, y_{\cap\{1,2\}}, y_{\cup\{1,2\}}) \\
& = \sum_{A_1, A_2 \subset E} (x_1 - 1)^{\rho(E)-\rho(A_1)} (x_2 - 1)^{\rho(E)-\rho(A_2)} \\
& \qquad (x_{\cap\{1,2\}} - 1)^{\rho(E)-\rho(A_1 \cap A_2)} (x_{\cup\{1,2\}} - 1)^{\rho(E)-\rho(A_1 \cup A_2)} \\
& \qquad (y_1 - 1)^{|A_1| - \rho(A_1)} (y_2 - 1)^{|A_2| - \rho(A_2)} \\
& \qquad (y_{\cap\{1,2\}} - 1)^{|A_1 \cap A_2| - \rho(A_1 \cap A_2)} (y_{\cup\{1,2\}} - 1)^{|A_1 \cup A_2| - \rho(A_1 \cup A_2)}.
\end{aligned}
$$

The Whitney rank generating function is defined as follows:

$$
R(M; x, y) = \sum_{A \subset E} x^{\rho(E)-\rho(A)} y^{|A| - \rho(A)}.
$$

For every matroid M, its Tutte polynomial and its Whitney rank generating function are equivalent under a simple change of variables:

$$R(M; x, y) = T(M; x+1, y+1).$$

We also naturally define the genus g Whitney rank generating function $R^{(g)}(M)$ as follows:

Definition 1.4. We will reuse the notations in the definition of the Tutte polynomial of genus g. By using these notations, let us define

$$\begin{aligned} R^{(g)}(M) &:= R^{(g)}(M; x_{\lambda_1}, y_{\lambda_1}, x_{\cap\lambda_2}, y_{\cap\lambda_2}, x_{\cup\lambda_2}, y_{\cup\lambda_2} : \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2) \\ &= T^{(g)}(M; x_{\lambda_1}+1, y_{\lambda_1}+1, x_{\cap\lambda_2}+1, y_{\cap\lambda_2}+1, x_{\cup\lambda_2}+1, y_{\cup\lambda_2}+1 : \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2). \end{aligned}$$

For every g-tuple $(A_i \mid i = 1, \dots, g)$ of subsets of E_M, $R^{(g)}(A_1, \dots, A_g)$ also can be defined naturally.

It is clear that $R^{(g)}(M^*; x, y) = R^{(g)}(M; y, x)$ holds for every positive integer g, again.

To state our results, we introduce the following concepts. If two matroids have the same Tutte polynomial for genus g, we call them $T^{(g)}$-equivalent. More generally, we introduce the following:

Definition 1.5. Let $\mathcal{M}$ be the set of all matroids. For $X \subset \mathcal{M}$, we call X is *of class* $T^{(g)}$ if $T^{(g)}$ is a complete invariant for X.

It is not difficult to see that each matroid $M = (E, \mathcal{I})$ can be reconstructed from its Tutte polynomial of genus $|E| + |\mathcal{B}(M)|$. Hence the set of Tutte polynomials $\{T^{(g)}\}_{g=1}^{\infty}$ is a complete invariant for matroids.

Here we state the following problem:

Problem 1.6. For $X \subset \mathcal{M}$, determine an appropriate small upper bound g such that X is of class $T^{(g)}$.

In the paper [11], the authors announced that each matroid M can be reconstructed from its Tutte polynomial of genus $|\mathcal{B}(M)|$ and that, for all g, there exist inequivalent matroids that have the same Tutte polynomial of genus g. For a real number r, $\lceil r \rceil$ denotes the minimum integer at least r. In this paper, we prove these statements.

Theorem 1.7. (1) *Every matroid M can be reconstructed from its Tutte polynomial of genus $|\mathcal{B}(M)|$. Especially, for $g \in \mathbb{N}$, the set of matroids $\{M \in \mathcal{M} \mid |\mathcal{B}(M)| \leq g\}$ is of class $T^{(g)}$.*

(2) *For any positive integer g, there exist two matroids M and N such that $T^{(g)}(M) = T^{(g)}(N)$ and $T^{(\lceil\sqrt{2}g\rceil)}(M) \neq T^{(\lceil\sqrt{2}g\rceil)}(N)$.*

This paper is organized as follows. In Section 2, we prepare some preliminaries. In Sections 3, 4 we give a proof of Theorem 1.7 (1) and (2), respectively. In section 5, we give some remarks and open problems.

All computations presented in this paper were performed using Magma [2] and Mathematica [15].

2 Preliminaries

2.1 Elementary Properties on Matroids

Definition 2.1. A matroid M is $T^{(g)}$-unique if M is a unique matroid, up to isomorphism, whose Tutte polynomial of genus g is $T^{(g)}(M)$.

Lemma 2.2. *If a matroid M has no circuit C with $3 \leq |C|$ then M is a direct sum of matroids of rank at most one. In particular, M is $T^{(1)}$-unique.*

Proof. Suppose that a matroid M has no circuit C with $3 \leq |C|$. If e is a loop or a coloop of M, then M can be decomposed into the direct sum of $M \setminus \{e\}$ and $\{e\}$. Now we assume here that M has neither a loop nor a coloop. Then every circuit C of M satisfies $|C| = 2$. In this case, every element e of E_M is contained in some circuit of two elements, and if e is contained in two distinct circuits $C_1 = \{e, x\}$ and $C_2 = \{e, y\}$ then the set $\{x, y\}$ is also a circuit of M. Hence if we denote $p \sim q$ for two elements p, q of E_M when $\{p, q\}$ is a circuit of M, then this relation $\sim$ is an equivalence relation on E_M. Each equivalence class of the above relation $\sim$ on E_M is corresponding to a uniform matroid of rank 1 and M can be decomposed into the direct sum of these uniform matroids. $\square$

Lemma 2.3 ([13, Proposition 1.1.6])**.** *Let $M = (E, \mathcal{I})$ be a matroid, $I \subseteq \mathcal{I}$ be an independent set of M, $e \in E \setminus I$ be an element outside of I. If $\rho(I \cup \{e\}) = |I|$ holds, then $I \cup \{e\}$ contains a unique circuit of M.*

Lemma 2.4. *Let C be a circuit of M with $|C| \geq 3$, and let B be a base of M such that $|C \setminus B| = 1$ holds. Then the following two conditions hold.*

1. *The family of bases contained in $B \cup C$ is $\mathcal{F} := \{(B \cup C) \setminus \{p\} | p \in C\}$.*

2. *If a base B' of M satisfies the condition that $\forall B \in \mathcal{F}, |B \setminus B'| \leq 1$, then B' is identical to a base in $\mathcal{F}$.*

Proof. The first condition follows immediately from Lemma 2.3. For the second condition, it is clear that $|C \setminus B'| = 1$, for otherwise there exist two distinct elements $u, v \in C \setminus B$ such that $\exists p \in C \setminus \{u, v\}, |B' \setminus ((B \cup C) \setminus \{p\})| \geq 2$, a contradiction. Then the second condition follows from the first condition. □

Lemma 2.5. *If a matroid M has a circuit of at least three elements, then, for every base B of M, there exists a circuit C with $|C| \geq 3$ such that $|C \setminus B| = 1$.*

Proof. Suppose not. Then, there exists a base B such that, for any element $e \in E_M \setminus B$, $\{e\} \cup B$ has a unique 2-circuit $\{e, e'\}$. Let $C = \{e_1, e_2, \ldots, e_m\}$ be a circuit of $M$of at least 3 elements. Let e'_i denote an element of B such that either $\{e_i, e'_i\}$ forms a 2-circuit of M or $e'_i = e_i$. Since the relation that two elements of E_M form a 2-circuit (are parallel) is an equivalence relation on E_M, the set $\{e'_1, e'_2, \ldots, e'_m\}$ is also a circuit of $M$of at least 3 elements. Nevertheless, the above circuit is a subset of B, a contradiction. □

Lemma 2.6. *Let B be a base of a non-separable matroid M and let C be a circuit of M such that $|C \setminus B| = 1$. If $|B \setminus C| > 0$ then M has another base B' such that $|C \setminus B'| = 1$ and $|B \setminus B'| \geq 2$ hold.*

Proof. If $|B \setminus C| > 0$ then, for every element $e \in B \setminus C$, $(E_M \setminus B) \cup \{e\}$ contains a unique cocircuit of at least 2 elements, and hence, there exists an element $e' \in E_M \setminus B$ such that $B'' := (B \setminus \{e\}) \cup \{e'\}$ is another base of M and $|C \setminus B''| = 1$ also holds. Then, for every element $e'' \in C \cap B'' \neq \emptyset$, $B' := (B'' \setminus \{e''\}) \cup (C \setminus B'')$ is a base of M such that $|C \setminus B'| = 1$ and $B \setminus B' = \{e, e''\}$ hold. □

2.2 Matroid Base Graph

First, let us define the matroid base graph of a matroid.

Definition 2.7 ([6, 8, 9])**.** Let M be a matroid, and let $\mathcal{B}(M)$ be the bases of M. Then the *matroid base graph* $\mathrm{G}(M) = (V(\mathrm{G}(M)), E(\mathrm{G}(M)))$ of M is defined by $V(\mathrm{G}(M)) := \mathcal{B}(M)$ and $E(\mathrm{G}(M)) := \{\{B, B'\} : B, B' \in \mathcal{B}(M), |B \setminus B'| = 1\}$.

For a vertex v of a graph G, its *open neighborhood* $N_G(v)$ is defined by the set $\{u \in V(G) : \{u, v\} \in E(G)\}$, and its *closed neighborhood* $N_G[v]$ is defined by the set $N_G(v) \cup \{v\}$.

Lemma 2.8 ([6, Lemma 4.1])**.** *Let M be a non-empty matroid and B a base of M. Then there are two partitions π and π' of $N_{\mathrm{G}(M)}(B)$ into non-void subsets so that*

1. *two vertices B_1, B_2 are adjacent in $N_{\mathrm{G}(M)}(B)$ if and only if B_1 and B_2 are in the same equivalence class of π or π', and*
2. *if $p \in \pi$ and $q \in \pi'$, then $|p \cap q| \leq 1$.*

Further, if M is non-separable, the pair of partitions π and π' are unique, up to order.

For any element $x \in B$, let $p_x^B := \{B' \in N_{\mathrm{G}(M)}(B) \mid x \notin B'\}$ and for any element $y \in E \setminus B$, let $q_y^B := \{B' \in N_{\mathrm{G}(M)}(B) \mid y \in B'\}$. Then both $\pi = \{p_x^B \mid x \in B \text{ and } p_x^B \neq \emptyset\}$ and $\pi' = \{q_y^B \mid y \in E \setminus B \text{ and } q_y^B \neq \emptyset\}$ are the partitions with the desired properties mentioned in Lemma 2.8. We will denote the former partition by $\pi(B, M)$ and the latter partition by $\pi'(B, M)$.

Combining the above fact with Lemma 2.8, we have the following:

Lemma 2.9 ([6])**.** *Let M be a matroid, B an arbitrary vertex of $\mathrm{G}(M)$. Suppose that we do not know the whole graph $\mathrm{G}(M)$, while we know only its subgraph induced by $N_{\mathrm{G}(M)}[B]$. Furthermore, suppose that we also know each of the two partitions of $N_{\mathrm{G}(M)}(B)$ corresponding to $\pi(B, M)$ and $\pi'(B, M)$. Then we can construct a labeling on the bases corresponding to the vertices $N_{\mathrm{G}(M)}[B]$, that is, we can recover the bases of M corresponding to $N_{\mathrm{G}(M)}[B]$.*

The above lemma is essentially proven in the paper [6]. Here we give its proof, only for the convenience of readers.

Proof. Let us treat each vertex in $N_{\mathrm{G}(M)}[B]$ as a base of M. First let us set $B := \{x_1, \ldots, x_{\rho(M)}\}$ and $E_M \setminus B := \{y_1, \ldots, y_{\rho(M^*)}\}$. From Lemma 2.4, we have that, for each element $y \in E_M \setminus B$, the subset $B \cup (\cup_{B' \in q_y^B} B')$ of

E_M turns out to be $B \cup \{y\}$. Furthermore, from Lemma 2.2, this subset $B \cup \{y\} = B \cup (\cup_{B' \in q_y^B} B')$ of E_M contains a unique circuit of M. Let us denote this circuit by C_y. In the same way, for every element $x \in B$, let us denote the unique cocircuit in $(E_M \setminus B) \cup \{x\}$ by D_x. For every vertex B'' in $N_{\mathrm{G}(M)}(B)$, we can find a unique pair of elements $(x(B''), y(B'')) \in \{(x, y) | x \in B, y \in E_M \setminus B\}$ such that $\{B''\} = p^B_{x(B'')} \cap q^B_{y(B'')}$. This fact reveals that $B'' = (B \setminus \{x(B'')\}) \cup \{y(B'')\}$. We also know that $\{x(B''), y(B'')\} = D_{x(B'')} \cap C_{y(B'')}$. Hence we can fix the labels on all the bases $N_{\mathrm{G}(M)}[B]$. In addition, by using the labels on the bases $N_{\mathrm{G}(M)}[B]$, we can also fix all the labels on the circuits $C_y (\forall y \in E_M \setminus B)$ and the cocircuits $D_x (\forall x \in B)$ of M. □

In general, matroids are not necessarily determined from their matroid base graphs. For example, every matroid M and its dual M^* have their common matroid base graph. More precisely, it is known that the matroid base graphs $\mathrm{G}(M_1)$ and $\mathrm{G}(M_2)$ are isomorphic as a graph if and only if M_1 and M_2 are equivalent ([6, Theorem 5.3]). Nevertheless, the following facts are known.

Let p, q are two vertices of a graph G, then $\mathrm{dist}(p, q)$ denotes the distance of p and q in G.

Lemma 2.10 ([6, Lemma 3.2])**.** *Let M be a matroid. If B and B' are two vertices of $\mathrm{G}(M)$ such that $\mathrm{dist}(B, B') = 2$, then $N_{\mathrm{G}(M)}(B) \cap N_{\mathrm{G}(M)}(B')$ is an induced subgraph of the cycle of four vertices and contains two nonadjacent vertices. That is, the intersection of the open neighborhoods is either two isolated vertices, a path of three vertices, or a cycle of four vertices.*

Lemma 2.11 ([6, Corollary 3.2.1])**.** *Let M be a matroid and B, B' be two vertices of $\mathrm{G}(M)$ such that $\mathrm{dist}(B, B') = 2$. Let B_1 and B_2 be nonadjacent vertices of $N_{\mathrm{G}(M)}(B) \cap N_{\mathrm{G}(M)}(B')$. Then if the labels on B, B_1 and B_2 are known, the labels on B' can be determined by $B' = (B_1 \cap B_2) \cup (B_1 \setminus B) \cup (B_2 \setminus B)$.*

By using Lemmata 2.8, 2.9, 2.10, and 2.11, C.A.Holzmann, P.G. Norton and M.D. Tobey [6] prove the following:

Lemma 2.12 ([6, Corollary 3.2.2])**.** *Let M be a matroid, and let B be a vertex of $\mathrm{G}(M)$. For every nonnegative integer i, let us define $V_B(i) := \{B' \in V(\mathrm{G}(M)) | \mathrm{dist}(B, B') \leq i\}$. Suppose that we do not know the whole graph $\mathrm{G}(M)$, while we know only its subgraph $\mathrm{G}(M)[V_B(i)]$ for some nonnegative integer i. Furthermore, suppose that we also know each of the two* *partitions of $N_{\mathrm{G}(M)}(B)$ corresponding to $\pi(B, M)$ and $\pi'(B, M)$. Let us set $B := \{x_1, \ldots, x_{\rho(M)}\}$ and $E_M \setminus B := \{y_1, \ldots, y_{\rho(M^*)}\}$. In this case, the labels on all vertices of $\mathrm{G}(M)[V_B(i)]$ can be determined.*

3 Proof of Theorem 1.7 (1)

We note that the number of the loops and the coloops is known from $T^{(1)}$. So we assume that there exists neither a loop nor a coloop. Furthermore, we also assume that our matroid M is non-separable simply because each $T^{(g)}$ (resp. $R^{(g)}$) factorizes into a product of the genus g Tutte polynomials (resp. the genus g Whitney rank generating functions) of non-separable matroids. Now we show the following statement equivalent to Theorem 1.7:

Theorem 3.1. *Every non-separable matroid M can be reconstructed from its Whitney rank generating function of genus $|\mathcal{B}(M)|$.*

Proof. First, we show that the Whitney rank generating function of genus $|\mathcal{B}(M)|$ constructs the matroid bases graph $\mathrm{G}(M)$. Then we show that the Whitney rank generating function of the genus $|\mathcal{B}(M)|$ reconstructs the matroid M itself.

1. Let $M = (E, \mathcal{I})$ be a matroid with g bases. We claim that $R^{(g)}(M)$ constructs $\mathrm{G}(M)$.

   In this case, $R^{(g)}(M)$ has a monomial $R^{(g)}(A_1, \ldots, A_g)$ such that each set $A_i (i = 1, \ldots, g)$ corresponds one-to-one to each base of M and it can be detected as follows. For a monomial of $R^{(g)}(M)$, it is clear that each set A_i is a base of M if and only if the variables x_{A_i} and y_{A_i} vanish (their exponents are zero) in the monomial. In addition, it is also clear that two bases $A_i, A_j (i \neq j)$ are mutually distinct if and only if the exponent of the variable $x_{A_i \cap A_j}$ is positive. By using this monomial $R^{(g)}(A_1, \ldots, A_g)$, we can obtain a graph $G = (V(G), E(G))$ isomorphic to the matroid base graph $\mathrm{G}(M)$ such that $V(G) := \{A_1, \ldots, A_g\}$ and $E(G) := \{\{A_i, A_j\} |$ the exponent of $x_{A_i \cap A_j}$ in $R^{(g)}(A_1, \ldots, A_g)$ is $1\}$.

2. Let $M = (E, \mathcal{I})$ be a non-separable matroid. As we mentioned in Section 1, the Tutte polynomial $T^{(1)}(M)$ gives the following numbers:

   - $\nu := |E|$

- $\rho :=$ the rank of M
- $\chi :=$ the corank of M
- $|\mathcal{B}(M)|$, the number of bases of M.

By Lemma 2.2, we can assume that M has a circuit of at least three elements. The matroid M has two distinct bases B_1, B_2 such that the distance $\mathrm{dist}(B_1, B_2)$ between these two vertices B_1, B_2 of $\mathrm{G}(M)$ coincides with the diameter $\mathrm{diam}(\mathrm{G}(M))$ of the graph $\mathrm{G}(M)$. Here we can assume that the number $\mathrm{diam}(\mathrm{G}(M))$ is at least 2, for otherwise, combining Lemmata 2.4 and 2.5, we have $\chi = 1$, and M turns out to be a single circuit, that is, $M = U_{k-1,k}$ for some positive integer k, which is $T^{(1)}$-unique. Hence we assume that $\mathrm{dist}(B_1, B_2) \geq 2$.

From Lemma 2.5, there exists a circuits C_1 of M such that $|C_1| \geq 3$ and $|C_1 \setminus B_1| = 1$ hold. Let c_1 be the unique element in $C_1 \setminus B_1$.

Let $g := |\mathcal{B}(M)|$. Then $R^{(g)}(M)$ has a monomial $R^{(g)}(A_1, \ldots, A_g)$ such that $A_1 = C_1$, $A_2 = B_1$, and $\{A_3, \ldots, A_{|C_1|+1}\} = q_{c_1}^{B_1} := \{B \in N_{\mathrm{G}(M)}(B_1) \mid c_1 \in B\}$ and $\{A_2, \ldots, A_g\} = \mathcal{B}(M) \setminus \{B_2\}$ hold. We guess this monomial $R^{(g)}(A_1, \ldots, A_g)$ of $R^{(g)}(M)$. That is, we take each monomial of $R^{(g)}(M)$ one by one, consider the monomial as if it were the monomial $R^{(g)}(A_1, \ldots, A_g)$, and check whether this candidate satisfies all of the properties and conditions described in the proofs that follow. If we find that the candidate at hand does not satisfy even one of these properties and conditions, we discard it.

Since $|C_1| > 2$, we have $A_1 \cup A_2 = A_2 \cup A_3 = A_3 \cup A_4 = A_4 \cup A_2 = \cup_{2 \leq i \leq |C_1|+1} A_i$.

Summarizing, we obtain $A_1, \ldots, A_g$ with a positive integer $m (\geq 3)$ such that

- $A_2, \ldots, A_g$ are mutually distinct bases of M. We can confirm this property by checking that the exponents of the variables $x_{A_i}, y_{A_i} (i = 2, \ldots, g)$ are zero and the exponents of the variables $x_{A_i \cap A_j} (2 \leq i < j \leq g)$ are positive in our candidate monomial.
- A_1 is a circuit of M of m elements such that $|A_1 \setminus A_i| = 1 (i = 2, \ldots, m)$ hold. We can confirm this property by checking that, for $2 \leq i \leq m$, the exponents of the variables x_{A_1} and $x_{A_1 \cap A_i}$ are all $\rho - m + 1$, the exponent of the variable y_{A_1} is 1, and the exponents

of the variables $y_{A_1 \cap A_i}$ are all 0 (c.f. Lemma 2.4) in our candidate monomial.

- Let H be a graph such that $V(H) := \{A_i | 2 \leq i \leq g\}$ and $E(H) := \{\{A_i, A_j\}|$ the exponent of $x_{A_i \cap A_j}$ in $R^{(g)}(A_1, \ldots, A_g)$ is $1\}$. Let $\mathrm{G}(M) - B_2$ denote the resulting subgraph obtained by deleting the vertex B_2 from $\mathrm{G}(M)$. Then there exists a graph isomorphism $f : V(\mathrm{G}(M) - B_2) \to V(H)$ such that $f(B_1) = A_2$ holds.

Let S denote the unique missing base in $\mathcal{B}(M) \setminus \{A_2, \ldots, A_g\}$.

From now on, we will show that we can reconstruct a correct labeling on all the bases $\mathcal{B}(M)$.

Since our graph H is isomorphic to the subgraph $\mathrm{G}(M) - B_2$, we can treat this graph H as if it is $\mathrm{G}(M) - B_2$. From Lemma 2.8, there is a unique pair $\{\pi, \pi'\}$ of partitions of $N_H(A_2)$ and only one of the two partitions (that is, $\pi'(A_2, M)$) has the equivalence class $\{A_3, \ldots, A_{m+1}\}$. Thus we can detect which of the two partitions of $N_H(A_2)$ is $\pi'(A_2, M)$. Combining the above with Lemmata 2.9, 2.10, 2.11 and 2.12, we can recover the labels on all bases $A_2, \ldots, A_g$ up to permutation of indices.

From now on, we will detect the labels on the unique missing base S of M on the outside of $V(H)$. From Lemma 2.5, M has a circuit C of at least 3 elements such that $|B \cup C| = \rho + 1$ holds. Since $|C| > 2$, from Lemma 2.4, $B \cup C$ contains (at least) 3 mutually distinct bases, namely, $S, A_{\sigma(1)}$ and $A_{\sigma(2)}$. In order to detect the labels on the circuit C, first we find candidates of the pair $\{A_{\sigma(1)}, A_{\sigma(2)}\}$ of distinct bases different from S, as follows: By using the above labeling, we enumerate the families $\{\mathcal{X}_i\}$ of labeled bases such that $\mathcal{X}_i = \{A_{\sigma_i(1)}, A_{\sigma_i(2)}, \ldots, A_{\sigma_i(k_i)}\}$, $| \cup_{j \leq k_i} A_{\sigma_i(j)}| = \rho + 1$, $\forall A_i \in \{A_2, \ldots, A_g\} \setminus \{A_{\sigma_i(1)}, \ldots, A_{\sigma_i(k_i)}\}, |A_i \cup (\cup_{j \leq k_i} A_{\sigma_i(j)})| \geq \rho + 2$. That is, $\mathcal{X}_i$ denotes the family of labeled bases of $V(H)$ included in the set $A_{\sigma_i(1)} \cup A_{\sigma_i(2)}$. Let $C(\mathcal{X}_i) := \cup_{j \leq k_i}((A_{\sigma_i(1)} \cup A_{\sigma_i(2)}) \setminus A_{\sigma_i(j)})$. Then either (1): M has a circuit $C'$of at least 3 elements such that $C(\mathcal{X}_i) = C' \cap S$ holds, or (2): The set $C(\mathcal{X}_i)$ is a circuit of M of k_i elements.

In the list of families $\{\mathcal{X}_i\}$, we can find a family $\mathcal{X}_j$ in the former case (1), as follows: M has a circuit C' of at least 3 elements that satisfies $C(\mathcal{X}_j) = C' \cap S$, if and only if one of the following two mutually exclusive conditions holds.

(a) If the set $A_{\sigma_j(1)} \cup A_{\sigma_j(2)}$ is a circuit of M, we can verify this fact by checking that the three conditions $|A_{\sigma_j(1)} \cup A_{\sigma_j(2)}| = k_j + 1$, $| \cap_{i=1}^{k_j} A_{\sigma_j(i)}| = 1$ and $k_j = \rho$ hold. In this case, we have that $S = (A_{\sigma_j(1)} \cup A_{\sigma_j(2)}) \setminus (\cap_{i=1}^{k_j} A_{\sigma_j(i)})$.

(b) Otherwise, from Lemmata 2.4 and 2.6, there exists $\mathcal{X}_\ell (\ell \neq j)$ such that $C(\mathcal{X}_j) \subsetneq C(\mathcal{X}_\ell)$ and $|C(\mathcal{X}_\ell) \setminus C(\mathcal{X}_j)| = 1$. In this case, we have that $S = (A_{\sigma_j(1)} \cup A_{\sigma_j(2)}) \setminus (C(\mathcal{X}_\ell) \setminus C(\mathcal{X}_j))$.

Hence, in either case, we can detect the labels on the missing base S. This completes the proof.

□

4 Proof of Theorem 1.7 (2)

4.1 Matroids M, N such that $T^{(1)}(M) = T^{(1)}(N)$ and $T^{(2)}(M) \neq T^{(2)}(N)$

In [7], the concept of "matroid relaxation" was given. Given a matroid $M = (E, \mathcal{B})$ with a subset $X \subset E$ that is both a circuit and a hyperplane, we can define a new matroid M' as the matroid with basis $\mathcal{B}' = \mathcal{B} \cup \{X\}$. That M' is indeed a matroid is easy to check. The Tutte polynomial of M' can be computed easily from the Tutte polynomial of M by

$$T(M'; x, y) = T(M; x, y) - xy + x + y. \tag{4.1}$$

In this subsection, E denote the set $\{z \mid z \in \mathbb{Z}, 1 \leq z \leq 2n\}$. For an independence system M, let $\mathcal{B}(M)$ denote the set of maximal independent set. In this subsection, let us define $X_1 := \{z \mid z \in \mathbb{Z}, 1 \leq z \leq n\}$, $X_2 := \{z \mid z \in \mathbb{Z}, n+1 \leq z \leq 2n\}$, $X_3 := (X_2 \cup \{n\}) \setminus \{2n\}$.

Let R_{2n} denote the independence system on E such that $\mathcal{B}(R_{2n}) = \mathcal{B}(U_{n,2n}) \setminus \{X_1, X_2\}$, and let Q_{2n} denote the independence system on E such that $\mathcal{B}(Q_{2n}) = \mathcal{B}(U_{n,2n}) \setminus \{X_1, X_3\}$.

Then, the following two facts can be easily proved.

Fact 4.1. For any positive integer n, R_{2n} is a matroid.

Fact 4.2. The independent system Q_4 is not a matroid since the application of the independent set exchange property to independent sets $\{\{1,3\},\{2\}\}$ forces one of $\{1,2\}$ and $\{2,3\}$ to be independent. However, Q_{2n} is a matroid if $n \geq 3$. Note that the dual matroid of R_{2n} is itself, and the dual matroid of Q_{2n} is isomorphic to Q_{2n}.

Therefore, $n \geq 3$ is a natural assumption in the following propositions.

Proposition 4.3. *For any integer $n \geq 3$, $T^{(1)}(R_{2n}) = T^{(1)}(Q_{2n})$ holds.*

Proof. Since we obtain $U_{n,2n}$ by two relaxations from both of R_{2n} and Q_{2n}, Therefore, the formula (4.1) yields

$$\begin{aligned} & T^{(1)}(R_{2n}; x, y) \\ = & T^{(1)}(Q_{2n}; x, y) \\ = & T^{(1)}(U_{n,2n}) + 2(xy - x - y) \\ = & \sum_{i=0}^{n} \binom{2n}{i} (x-1)^{n-i} + \sum_{i=n+1}^{2n} \binom{2n}{i} (y-1)^{i-n} + 2(xy - x - y). \square \end{aligned}$$

Proposition 4.4. *For any integer $n \geq 3$, $T^{(2)}(R_{2n}) \neq T^{(2)}(Q_{2n})$ holds.*

Proof. Let (A, B) be a pair of sets such that $A \cap B = \emptyset, |A| = |B| = n$ and $\rho(A) = \rho(B) = n - 1$ hold. The matroid R_{2n} has such a pair while Q_{2n} does not. Hence the Tutte polynomial $T^{(2)}(R_{2n})$ have the term

$$(x_1 - 1)^{n-1}(y_1 - 1)(x_2 - 1)^{n-1}(y_2 - 1)(x_{\cap(1,2)} - 1)^n(y_{\cup(1,2)} - 1)^n,$$

which is generated by $(A, B) = (X_1, X_2)$. Since $T^{(2)}(Q_{2n})$ does not have this term, we have $T^{(2)}(R_{2n}) \neq T^{(2)}(Q_{2n})$. $\square$

These two propositions prove Theorem 1. 1 (2).

Remark 4.5. We can calculate the proper small difference

$$\begin{aligned} & T^{(2)}(R_{2n}; x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) - T^{(2)}(Q_{2n}; x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \\ = & 2(x_1y_1 - x_1 - y_1)(x_2y_2 - x_2 - y_2)(x_3y_4 - x_3 - y_4)(x_3 - 1)^{n-1}(y_4 - 1)^{n-1}, \end{aligned}$$

where we use x_3, x_4, y_3 and y_4 instead of $x_{\cap(1,2)}$, $x_{\cup(1,2)}$, $y_{\cap(1,2)}$ and $y_{\cup(1,2)}$ in short.

The calculation is not difficult, but very long. Therefore, we leave the proof in the Appendix.

4.2 Non-isomorphic matroids which need higher genus

In this subsection, we always assume that the ground set E of our matroids is the cyclic group $\mathbb{Z}_{4n}$, where $n \geq 3$. Moreover, when there is no fear of confusion, the cyclic group $\mathbb{Z}_{4n}$ is identified as the set of integers $\{z \mid 0 \leq z \leq 4n-1\}$.

Definition 4.6. Let $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ be the families of subsets of E defined by

$$\begin{aligned}
\mathcal{F}_1 &:= \{\{2i, 2i+1, 2i+2\} \mid 0 \leq i \leq n-2\} \cup \{\{2n-2, 2n-1, 0\}\},\\
\mathcal{F}_2 &:= \{\{2i, 2i+1, 2i+2\} \mid n \leq i \leq 2n-2\} \cup \{\{4n-2, 4n-1, 2n\}\},\\
\mathcal{F}_3 &:= \{\{2i, 2i+1, 2i+2\} \mid 0 \leq i \leq n-1\},\\
\mathcal{F}_4 &:= \{\{2i, 2i+1, 2i+2\} \mid n \leq i \leq 2n-1\}.
\end{aligned}$$

We say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are *circuital loose cycles* of length n, and $\mathcal{F}_3$ and $\mathcal{F}_4$ are *circuital loose paths* of length n. Let S_{4n} and S'_{4n} be the independent systems as follows;

$$\begin{aligned}
\mathcal{B}(S_{4n}) &= \mathcal{B}(U_{3,4n}) \setminus (\mathcal{F}_1 \cup \mathcal{F}_2);\\
\mathcal{B}(S'_{4n}) &= \mathcal{B}(U_{3,4n}) \setminus (\mathcal{F}_3 \cup \mathcal{F}_4).
\end{aligned}$$

Fact 4.7. Notice that S'_4 is not a matroid because the independent sets $\{0, 2\}$ and $\{0, 1, 3\}$ force one of $\{0, 1, 2\}$ and $\{2, 3, 0\}$ to be independent. For the same reason, S_8 is not a matroid. Therefore, the assumption $n \geq 3$ is essential.

Remark 4.8. The odd elements of E are essential for the same reason that Q_4 does not satisfy the axioms of independent sets.

Fact 4.9. Let M be one of S_{4n} and S'_{4n}, and $y, z \in \mathbb{Z}_{4n}$ be elements. Then, the number of circuits of three elements containing both y and z is at most one.

Proof. This fact can be easily checked. □

Proposition 4.10. *The independent systems S_{4n} and S'_{4n} are matroids.*

Proof. Let M be one of the independent systems S_{4n} and S'_{4n}. Let A_1, A_2 be independent subsets of M on E such that $|A_2| < |A_1| \leq 3$. If $|A_2| \leq 1$, then there exists an element $x \in A_1 \setminus A_2$ such that $A_2 \cup \{x\}$ is independent because all sets of size at most two are independent in both S_{4n} and S'_{4n}. Therefore, we can assume $|A_2| = 2$ and $|A_1| = 3$. If there does not exist a circuit of size 3 which contains A_2, then $A_2 \cup \{x\}$ is independent for each element $x \in A_1 \setminus A_2$. Therefore, we consider the case that A_2 is contained in a circuit of size 3. Let us assume that $\{z_1, z_2, z_3\}$ is a circuit, and that $A_2 = \{z_1, z_2\}$. Because A_1 is independent, A_1 contains an element $x \notin \{z_1, z_2, z_3\}$. Then, $A_2 \cup \{x\}$ is independent from Fact 4.9. □

Fact 4.11. If a subset $A \subset E$ is none of $\{\{2n-2, 2n-1, 0\}, \{2n-2, 2n-1, 2n\}, \{4n-2, 4n-1, 0\}, \{4n-2, 4n-1, 2n\}\}$, then $\rho_{S_{4n}}(A) = \rho_{S_{4n}}(A+2n) = \rho_{S'_{4n}}(A) = \rho_{S'_{4n}}(A+2n)$ holds.

Fact 4.12. For a subset $A \ni 0$ of E, $\rho_{S_{4n}}(A) = \rho_{S'_{4n}}(((A+2n) \setminus \{2n\}) \cup \{0\})$

Theorem 4.13. *For any $g \in \mathbb{N}$, there exists a pair of matroids (M, M') such that $T^{(g)}(M) = T^{(g)}(M')$ and $M \not\simeq M'$.*

Proof. For a g-tuple $\mathcal{A} = (A_1, A_2, \ldots, A_g)$ $(A_i \subset E)$, define a family $\mathcal{X}(A)$ as follows;

$$\mathcal{X}(\mathcal{A}) := \{A_i\}_{i=1}^{g} \cup \{A_i \cup A_j\}_{1 \le i < j \le g} \cup \{A_i \cap A_j\}_{1 \le i < j \le g}.$$

We fix a positive integer n such that $n > g^2 = |\mathcal{X}(\mathcal{A})|$.

Let l and l' be the non-negative integer defined as follows; $l := 0$ if neither $\{2n-2, 2n-1, 0\}$ nor $\{2n-2, 2n-1, 2n\}$ belongs to $\mathcal{X}(\mathcal{A})$, $l := \min\{i \mid \{\{2(n-i), 2(n-i)-1, 2(n-i-1)\} \notin \mathcal{X}(\mathcal{A})$ otherwise; $l' := 0$ if neither $\{4n-2, 4n-1, 0\}$ nor $\{4n-2, 4n-1, 2n\}$ belongs to $\mathcal{X}(\mathcal{A})$, $l' := \min\{i \mid \{\{2(2n-i), 2(2n-i)-1, 2(2n-i-1)\} \notin \mathcal{X}(\mathcal{A})$ otherwise. Here, we assume $l \ge l'$ because the other case can also be proved similarly. Moreover, the inequation $n > g^2$ yields $n > l$.

Here, we define a bijection on E as follows:

$$\phi_{\mathcal{A}}(i) = \begin{cases} k & (\text{if } 0 \le k \le 2(n-l)-1 \text{ or } 2n \le k \le 2(2n-l)-1), \\ k+2n & (\text{if } 2(n-l-1) \le k \le 2n-1) \text{ or } 2(2n-l-1) \le k \le 4n-1) \\ k-2n & (\text{if } 2(2n-l-1) \le k \le 4n-1). \end{cases}$$

Since the map $\phi_{\mathcal{A}}$ is bijective for any family $\mathcal{A}$, hence, for any set $R \subset E$, we have $|\phi_{\mathcal{A}}(R)| = |R|$. Furthermore, for the case that R is none of the four sets $\{2n-2, 2n-1, 0\}$, $\{2n-2, 2n-1, 2n\}$, $\{4n-2, 4n-1, 0\}$,

$\{4n-2, 4n-1, 2n\}$, the equation $\rho_{S'_{4n}}(\phi_{\mathcal{A}}(R)) = \rho_{S_{4n}}(R)$ clearly holds. For the case that R is one of the four sets $\{2n-2, 2n-1, 0\}$, $\{2n-2, 2n-1, 2n\}$, $\{4n-2, 4n-1, 0\}$, $\{4n-2, 4n-1, 2n\}$, we have:

$$\begin{aligned}
\rho_{S_{4n}}(\{2n-2, 2n-1, 0\}) &= 2, & \rho_{S'_{4n}}(\{4n-2, 4n-1, 0\}) &= 2,\\
\rho_{S_{4n}}(\{4n-2, 4n-1, 2n\}) &= 2, & \rho_{S'_{4n}}(\{2n-2, 2n-1, 2n\}) &= 2,\\
\rho_{S_{4n}}(\{2n-2, 2n-1, 2n\}) &= 3, & \rho_{S'_{4n}}(\{4n-2, 4n-1, 2n\}) &= 3,\\
\rho_{S_{4n}}(\{4n-2, 4n-1, 0\}) &= 3, & \rho_{S'_{4n}}(\{2n-2, 2n-1, 4n\}) &= 3.
\end{aligned}$$

Then, for any i, j, we have

$$\begin{cases}
|A_i| = |\phi_{\mathcal{A}}(A_i)|,\\
\rho_{S_{4n}}(A_i) = \rho_{S'_{2n}}(\phi_{\mathcal{A}}(A_i)),\\
|A_i \cap A_j| = |\phi_{\mathcal{A}}(A_i) \cap \phi_{\mathcal{A}}(A_j)|,\\
\rho_{S_{4n}}(A_i \cap A_j) = \rho_{S'_{2n}}(\phi_{\mathcal{A}}(A_i) \cap \phi_{\mathcal{A}}(A_i)),\\
|A_i \cup A_j| = |\phi_{\mathcal{A}}(A_i) \cup \phi_{\mathcal{A}}(A_j)|,\\
\rho_{S_{4n}}(A_i \cup A_j) = \rho_{S'_{2n}}(\phi_{\mathcal{A}}(A_i) \cup \phi_{\mathcal{A}}(A_i)).
\end{cases}$$

Therefore, we obtain the same monomial in $R^{(g)}(M)$ and $R^{(g)}(M')$. $\square$

Suppose that, for any g-tuple $\mathcal{A}$, the family $\mathcal{X}(\mathcal{A})$ contains neither a circuital loose cycle nor a loose path of length n. Then, from Theorem 4.13, we have that $T^{(g)}(S_{4n}) = T^{(g)}(S'_{4n})$. In other words, if $T^{(g)}(S_{4n}) \neq T^{(g)}(S'_{4n})$ holds for some positive integer g, then we have a g-tuple $\mathcal{A}$ such that the family $\mathcal{X}(\mathcal{A})$ contains either a circuital loose cycle or a loose path of length n.

Using the above observation with some detailed calculations, we obtain a considerably better lower bound value as follows:

Proposition 4.14. *If $g \leq \left\lceil \frac{5+\sqrt{8n-15}}{2} \right\rceil$ then $T^{(g)}(S_{4n}) = T^{(g)}(S'_{4n})$.*

In order to prove the above, we use the following two lemmata.

Lemma 4.15. *Let g be a positive integer such that $T^{(g)}(S_{4n}) \neq T^{(g)}(S'_{4n})$. And let $\mathcal{A} = (A_1, A_2, \ldots, A_g)$ $(A_i \subset E)$ be a g-tuple such that $\mathcal{X}(\mathcal{A})$ contains either a circuital loose cycle or loose path of length n. Suppose that there exist two distinct indices $i \neq j$ such that $A_i \cup A_j$ is a circuit of size 3 which belongs to a circuital loose cycle or loose path of length n. Then we have another family $\mathcal{A}' = (A'_1, A'_2, \ldots, A'_{g-1})$ $(A'_i \subset E)$ such that $\mathcal{X}(\mathcal{A}')$ contains either a circuital loose cycle or loose path of length n.*

Proof. Let $A_i \cup A_j (i \neq j)$ be a circuit of size 3 which belongs to a circuital loose cycle or loose path of length n. If $A_i \subset A_j$, then we have $\mathcal{A}' := \mathcal{A} \setminus \{A_i\}$. Otherwise, we have $\mathcal{A}' := (\mathcal{A} \setminus \{A_i, A_j\}) \cup \{A_i \cup A_j\}$ because none of A_i, A_j, $A_i \cap A_k$, and $A_j \cap A_k$ $k \notin \{i, j\}$ is a circuit of size 3. $\square$

Lemma 4.16. *Let n be a positive integer at least 4 and let g be a positive integer such that $T^{(g)}(S_{4n}) \neq T^{(g)}(S'_{4n})$. And let $\mathcal{A} = (A_1, A_2, \dots, A_g)$ $(A_i \subset E)$ be a g-tuple such that $\mathcal{X}(\mathcal{A})$ contains either a circuital loose cycle or loose path of length n. Then, there exist $\{A'_i \subset E\}_{i=1}^{g}$ such that we can construct either a circuital loose cycle or path of length n by using the set of hyper-edges $\{A'_i \cap A'_j \mid 3 \leq i < j \leq g\} \cup \{A'_1, A'_2\}$.*

Proof. This proof is divided into several subcases as follows:
(Case 1) If $\mathcal{F}_1$ does not contain $A_i \cap A_j$ and $\{A_i\}_{i=1}^{g} = \mathcal{F}$, then renumbering of the indices of $\{A_i\}$ we assume that $A_1 = \{0, 1, 2\}$, $A_2 = \{2, 3, 4\}$, $\dots$, $A_{n-1} = \{2n-4, 2n-3, 2n-2\}$, $A_n = \{2n-2, 2n-1, 0\}$.

Then setting $A'_i := A_i \cup A_{i+1} = \{2i, 2i+1, 2i+2, 2i+3, 2i+4\}$ for each $i \in \mathbb{Z}_n$, we obtain $A'_{i-1} \cap A'_i = \{2i, 2i+1, 2i+2\}$.

Thus, from now on, we assume that there exists a pair of indices $(i, j)(i \neq j)$ such that $A_i \cap A_j \in \mathcal{F}_1$.

(Case 2) Let $l \geq 4$ and let us assume without loss of generality that $A_1 \cap A_2 = \{0, 1, 2\}$, $A_3 = \{2, 3, 4\}$, $A_4 = \{4, 5, 6\}$, $\dots$, $A_l = \{2(l-2), 2l-3, 2l-2\}$, $A_{l+1} \cap A_{l+2} = \{2l-2, 2l-1, 2l\}$. Then we redefine that $A_i := A_i \setminus \{z \mid 3 \leq z \leq 2l-3\}$ for $i \in \{1, 2, l+1, l+2\}$.
(Subcase 2-1) Let us assume that $l-1 = n$, i.e., $A_{l+1} \cap A_{l+2} = \{0, 1, 2\}$. Let us define that $A'_1 := \{z \mid 0 \leq z \leq 2n-1\}$ and $A'_i := A_i$ for $2 \leq i \leq l$. Then, the family $\mathcal{A}' := \{A'_i \mid 1 \leq i \leq l\}$ satisfies $\mathcal{F}_1 \subset \{A'_i \cap A'_1 \mid i \neq 1\}$. (Subcase 2-2) Let us assume that $A_2 \notin \{A_{l+1}, A_{l+2}\}$. Let us define that $A'_1 := A_1$, $A'_i := A_i \cup A_{i+1} = \{2(i-2), 2i-3, 2i-2, 2i-1, 2i\}$ for $2 \leq i \leq l-1$, and $A'_i := A_i$ for $i \in \{l, l+1, l+2\}$. Then, the family $\mathcal{A}' := \{A'_i \mid 1 \leq i \leq l\}$ satisfies $\{\{2i, 2i+1, 2i+2\} \mid 0 \leq i \leq l-1\} \subset \{A'_i \cap A'_j \mid i \neq j\}$.

(Case 3) Let l be an integer at least 3, and we assume that $\mathcal{F}_1 \subset \{A_i \cap A_j \mid 1 \leq i < j \leq g-l\} \cup \{B_i \mid g-l+1 \leq i \leq g\}$ and $B_i \cap B_j = \emptyset$ for each $i \neq j$. Let us define that $A'_i := A_i$ for $1 \leq \forall i \leq g-l$ and $B'_j := B_j \cup B_{j+1}$ (mod l). Then, $B_i \cap B_j \neq \emptyset$ if and only if $i - j \neq \pm 1$. Then, the family $\mathcal{A}' = \{A'_i\}_{i=1}^{g-l} \cup \{B'_j\}_{j=g-l+1}^{g}$ satisfies $\mathcal{F}_1 \subset \{A'_i \cap A'_j (i \neq j)\} \cup \{B'_i \cap B'_j (i \neq j)\}$.

From the proof so far, there exist a family $\{A_i\}_{i=1}^g$ and an integer $l \in \{0,1,2\}$ such that $\mathcal{F}_1 \subset \{A_i \cap A_j \mid 1 \leq i < j \leq g-l\} \cup \{A_i \mid g-l \leq i \leq g\}$. $\square$

Proof of Proposition 4.14. Solving the following inequality: $\binom{g-2}{2} + 2 \geq n$, we obtain the desired bound. $\square$

Theorem 3.1 says that $T^{(g)}(S_{4n}) \neq T^{(g)}(S_{4n})$ for for $g \geq \binom{4n}{3} - 2n$. However, this upper bound is too large.

Here, we introduce an invariant of finite simple graphs.

Definition 4.17. Let $G = (V(G), E(G))$ be a finite simple graph. We say that a family $\mathcal{H} = \{H_1, \ldots, H_s\}$ of subsets of V is a *k-fold intersecting cover* if, for each edge $\{u,v\} \in E(G)$, there exists a k-subset $X_{\{u,v\}}$ of the index set $[1,s]$ such that $\{u,v\} = \cap_{i \in X_{\{u,v\}}} H_i$. We say that a finite simple G is *k-fold intersecting coverable* by s sets if there exists a family $\mathcal{H} = \{H_1, H_2, \ldots, H_s\}$ of subsets of V such that $\mathcal{H}$ is a k-fold intersecting cover of G. Let $\iota_k(G)$ be the minimum number s such that G is k-fold intersecting coverable by s sets. We call this number $\iota_k(G)$ the k-fold intersecting cover number of G. For simplicity, only when $k = 2$, we use the symbol $\iota(G)$ instead of $\iota_2(G)$. Clearly, G has a 2-fold intersecting cover with $|E(G)|+1$ sets, those are the vertex set and all edges.

Let C_n denote a cycle graph whose edge set is $\{\{i, i+1\} \mid i \in \mathbb{Z}_n\}$, and P_{n+1} denote a path graph whose edge set is $\{\{i, i+1\} \mid 0 \leq i \leq n\}$.

Theorem 4.18. *The equation $\iota(C_n) = \iota(P_{n+1})$ holds.*

Proof. Let $\mathcal{H} = \{H_i\}_{i=1}^s$ be a 2-fold intersecting cover of P_{n+1}, and ψ be the ring isomorphism from $\mathbb{Z}$ to $\mathbb{Z}_n$. For a set $H_i \in \mathcal{H}$, let us define $H'_i := \{\psi(z) \mid z \in H\}$. Note that $|H'_i \cap H'_j| \leq |H_i \cap H_j|$ for each i, j. Then, if $H_i \cap H_j$ is an edge of P_{n+1}, $H'_i \cap H'_j$ is an edge of C_n. Therefore, $\mathcal{H}' := \{H'_i\}_{i=1}^s$ is a 2-fold intersecting cover of C_n.

Conversely, let $\mathcal{H} = \{H_i\}_{i=1}^s$ be a 2-fold intersecting cover of C_n such that $\{0,1\} = H_1 \cap H_s$, and ψ' be the natural injection from $\mathbb{Z}_n$ to $\{0, \ldots, n\}$ such that $\psi'(\bar{0}) = 0$ and $\psi'(\bar{i}) = i$ for each $\bar{i} \in \mathbb{Z}_n \setminus \{\bar{0}\}$. Then, we define the set H'_i as follows; (i) if $\{\overline{n-1}, \bar{0}, \bar{1}\} \subset H_i$, let $H'_i := \{\psi'(\bar{z}) \mid \bar{z} \in H_i\} \cup \{n\}$; (ii) if $\{\overline{n-1}, \bar{0}\} \subset H_i \not\ni \bar{1}$, let $H'_i := (\{\psi'(\bar{z}) \mid \bar{z} \in H_i\} \cup \{n\}) \setminus \{0\}$; (iii) otherwise, $H'_i := \{\psi'(\bar{z}) \mid \bar{z} \in H_i\}$. Then, if $H_i \cap H_j$ is an edge of C_n, $H'_i \cap H'_j$ is an edge of P_{n+1}. Therefore, $\mathcal{H}' := \{H'_i\}_{i=1}^s$ is a 2-fold intersecting cover of P_{n+1}. $\square$

Theorem 4.19. *The inequation $T^{\iota(C_n)}(S_{4n}) \neq T^{\iota(C_n)}(S'_{4n})$ holds.*

Proof. The Whitney rank generating function $R^{\iota(C_n)}(S_{4n})$ has a monomial generated by $\mathcal{F}_1$, while $R^{\iota(C_n)}(S'_{4n})$ does not. $\square$

Theorem 4.20. *For each integer at least* 3*, we have $\iota(C_n) \leq 2\lceil\sqrt{n}\rceil$.*

Proof. (Case 1) We consider the case that n is a square number at least 9, e.g., we assume $n = p^2$. Then, we have $p \geq 3$.

Take $2p$ sets $A_0, A_1, \ldots, A_{p-1}, B_0, B_1, \ldots, B_{p-1}$ as follows;

$$\begin{aligned} A_i &:= \{z \in \mathbb{Z}_{p^2} \mid ip \leq z \leq ip+1\}, \\ B_j &:= \bigcup_{0 \leq k \leq p-2} ((j+k+1)p + \{k+1, k+2\}) \end{aligned}$$

Because $A_{i+1} = A_i + p$ and $B_{j+1} = B_j + p$ hold for $0 \leq i, j \leq p-1$, the indices are taken modulo p.

It is clear that for $0 \leq i \leq p-1$, we have

$$A_i \cap A_{i+1} = \{(i+1)p, (i+1)p+1\}. \tag{4.2}$$

It is also clear that $A_0 \cap B_0 = \{p+1, 0\}$ from the definition $B_0 = \{p+1, p+2, , 2p+2, 2p+3, 3p+3, 3p+4, \ldots, (p-1)p+p-1, (p-1)p+p = 0\}$. Thus, we have $A_i \cap B_i = \{ip, (i+1)p+1\}$ from $A_i = A_{i-1} + ip$ and $B_i = B_{i-1} + ip$.

For $1 \leq r \leq p-1$, we have the following by substituting $j = q-p$ and $k = r-1 (\leq p-2)$;

$$\begin{aligned} A_q \cap B_{q-r} &\supset (((q-r) + (r-1) + 1)p + \{(r-1)+1, (r-1)+2\}) \\ &= \{pq+r, pq+r+1\}. \end{aligned}$$

If $r \neq p-1$, we have $\{p(q+1)+r+1, p(q+1)+r+2\} \subset A_{q+1} \cap B_{q-r}$ and $\{p(q+1)+r+1, p(q+1)+r+2\} \cap (A_q \cap A_{q+1}) = \emptyset$ from $r \geq 1$ and (4.2). If $r = p-1$, we have $B_{q-r} = B_{q+1}$, i.e., $A_{q+1} \cap B_{q+1} = \{(q+1)p, (q+2)p+1\}$ and $A_q \cap B_{q-r} = \{(q+1)p-1, (q+1)p\}$. Then, we have $p(q+2)+1 \notin A_q$ from (4.2). In both cases, the elements of $(A_{q+1} \cap B_{q-r}) \setminus \{pq+r, pq+r+1\}$ do not belong to A_q. By analogous argument, the elements of $(A_{q-1} \cap B_{q-r}) \setminus \{pq+r, pq+r+1\}$ do not belong to A_q. Therefore, we have the following equation;

$$A_q \cap B_{q-r} = \begin{cases} \{pq, p(q+1)+1\} & (\text{if } r = 0), \\ \{pq+r, pq+r+1\} & (\text{otherwise }). \end{cases} \tag{4.3}$$

Therefore, $p+1$ edges of A_q are $A_{q-1}\cap A_q = \{pq, pq+1\}$, $A_q\cap B_{q-1} = \{pq+1, pq+2\}$, $A_q\cap B_{q-2} = \{pq+2, pq+3\}$, $\ldots$, $A_q\cap B_{q+1} = \{p(q+1)-1, p(q+1)\}$ $A_q\cap A_{q-1} = \{p(q+1), p(q+1)+1\}$.

(Case 2) We consider the case that n is not a square number at least 12. Let p be the smallest integer such that $n < p^2$. Then we have $p \geq 4$. Let $\mathcal{H}^{(0)} = \{A_i^{(0)}, B_j^{(0)} \mid i, j \in \mathbb{Z}_p\} := \{A_i, B_j \mid i, j \in \mathbb{Z}_p\}$ be a 2-fold intersecting cover of $\mathbb{Z}_{p^2}$ in (Case 1). Define graphs $G^{(q)}$ $(1 \leq q \leq p)$ with vertex set $\mathbb{Z}_{p^2}$ recursively as follows; $G^{(0)} := C_{p^2}$, take an integer k_q such that $2 \leq k_q \leq p-1$, and define $G^{(q)}$ as $E(G^{(q)}) = (E(G^{(q-1)}) \setminus \{(i, i+1) \mid pq+1 \leq i \leq pq+k_q-1\}) \cup \{(pq+1, pq+k_q)\}$. Clearly, we have $|E(G^{(q)})| = |E(G^{(q-1)})| - (k_q-2)$. In particular, $|E(G^{(q)})| = |E(G^{(q-1)})| - (p-3)$ if $k_q = p-1$. For each $1 \leq q \leq p$, we define recursively $\mathcal{H}^{(q)} = \{A_i^{(q)}, B_{q-j}^{(q)} \mid i, j \in \mathbb{Z}_p\}$ as follows;

$$A_i^{(q)} := \begin{cases} A_i^{(q-1)} & (\text{if } i \neq q), \\ A_i^{(q-1)} \setminus \{pq+2, pq+3, \ldots, pq+k_q-1\} & (\text{if } i = q), \end{cases}$$

$$B_{q-j}^{(q)} := \begin{cases} B_{q-j}^{(q-1)} & (\text{if } k_q \leq j \leq p), \\ B_{q-j}^{(q-1)} \setminus \{pq+j, pq+j+1\} & (\text{if } 2 \leq j \leq k_q-1), \\ (B_{q-j}^{(q-1)} \setminus \{pq+2\}) \cup \{pq+k_q\} & (\text{if } j = 1). \end{cases}$$

Then, the following holds for each $1 \leq q \leq p$;

- $pq+2, pq+3, \ldots, pq+k_q \in A_q^{(q-1)} \setminus \bigcup_{i\neq q} A_i^{(q-1)}$ because of $2 \leq k_q \leq p-1$;
- $B_j^{(q)} \cap A_i^{(q)} = B_j^{(q-1)} \cap A_i^{(q-1)}$ holds for each j if $i \neq q$ because each element of $\mathbb{Z}_{p^2} \setminus A_q^{(q-1)}$ belongs to $A_i^{(q)}$ (resp. $B_j^{(q)}$) if and only if it belongs to $A_i^{(q-1)}$ (resp. $B_j^{(q-1)}$);
- $A_{i-1}^{(q)} \cap A_i^{(q)} = A_{i-1}^{(q-1)} \cap A_i^{(q-1)} = \{ip, ip+1\}$ also holds for each i;
- the edges in $A_q^{(q)}$ of $G^{(q)}$ are $A_{q-1}^{(q)} \cap A_q^{(q)} = \{pq, pq+1\}$, $A_q^{(q)} \cap B_{q-1}^{(q)} = \{pq+1, pk+k_q\}$, $A_q^{(q)} \cap B_{q-k_q}^{(q)} = \{pq+k_q, pq+k_q+1\}$, $\ldots$, $A_q^{(q)} \cap B_{q+1}^{(q)} = \{p(q+1)-1, p(q+1)\}$, and $A_q^{(q)} \cap A_{q+1}^{(q)} = \{p(q+1), p(q+1)+1\}$.

Then, $\mathcal{H}^{(q)}$ is a 2-fold intersecting cover of $G^{(q)}$. Therefore, C_n has a 2-fold intersecting cover with $2p$ sets if $3p \leq n < p^2$.

(Case 3) We consider the remaining small n. Then, the 2-fold intersecting covers

- $\{\{1,2\},\{2,0\},\{0,1\},\{0,1,2\}\}$ for the case of C_3,
- $\{\{i,i+1,i+2\} \mid i \in \mathbb{Z}_4\}$, for the case of C_4,
- $\{\{i,i+1,i+2\} \mid i \in \mathbb{Z}_5\}$ for the case of C_5,
- $\{\{i,i+1,i+2\} \mid i \in \mathbb{Z}_6\}$ for the case of C_6,
- $A_1 = \{0,1,2\}, A_2 = \{1,2,3,4\}, A_3 = \{3,4,5,6\}, A_4 = \{5,6,0,1\}, A_5 = \{2,3,6,0\}, A_6 = \{4,5\}$ for C_7,
- $A_1 = \{0,1,2,3\}, A_2 = \{2,3,4,5\}, A_3 = \{4,5,6,7\}, A_4 = \{6,7,0,1\}, A_5 = \{1,2,5,6\}, A_6 = \{3,4,7,0\}$ for C_8,
- $A_1 = \{0,1,2,3\}, A_2 = \{2,3,4,5\}, A_3 = \{4,5,6,7\}, A_4 = \{6,7,8,9\}, A_5 = \{8,9,0,1\}, A_6 = \{1,2,5,6\}, A_7 = \{3,4,7,8\}, A_8 = \{9,0\}$ for C_{10},
- $A_1 = \{0,1,2,3,4\}, A_2 = \{3,4,5,6\}, A_3 = \{5,6,7,8\}, A_4 = \{7,8,9,10\}, A_5 = \{9,10,0,1\}, A_6 = \{1,2,6,7\}, A_7 = \{2,3,8,9\}, A_8 = \{4,5,10,0\}$ for the case C_{11}

satisfy the statement of our theorem. □

Proposition 4.21. *For each integer $n \geq 3$, we have the following;*

$$2\left\lceil \sqrt{n} \right\rceil \leq \left\lceil \sqrt{2} \left\lceil \frac{5+\sqrt{8(n-2)+1}}{2} \right\rceil - 2 \right\rceil .$$

Let u_n, l_n, $\hat{l}_n$ be the following monotonically non-decreasing sequences such that

$$\begin{aligned} u_n &= 2\left\lceil \sqrt{n} \right\rceil , \\ l_n &= \left\lceil \frac{-1+\sqrt{8(n-2)+1}}{2} + 3 \right\rceil , \\ \hat{l}_n &= \left\lceil \sqrt{2}\, l_n \right\rceil - 2. \end{aligned}$$

Proof. If $n = 3$, it is trivial because $u_3 = \hat{l}_3 = 4$. Let us assume that $n \geq 4$. Let n' be the maximum square number at most n. It is well-known that $(l_{n'}-3)(l_{n'}-2)/2$ is the minimum triangular number at least $n'-2$. Therefore, we have

$$4n' \leq 2(l_{n'}-3)(l_{n'}-2)+8 = \left(\sqrt{2}l_{n'} - \frac{5}{\sqrt{2}}\right)^2 - \frac{5}{2} < \left\lceil \sqrt{2}l_{n'} - 3 \right\rceil^2.$$

Because the both sides of the above inequation are square numbers, we have

$$u_{n'} = 2\sqrt{n'} < \left\lceil \sqrt{2}l_{n'} \right\rceil - 3.$$

Because the both sides of the above inequation are positive integers, we have

$$u_{n'} \leq \left\lceil \sqrt{2}l_{n'} \right\rceil - 4 = \hat{l}_{n'} - 2.$$

If $n = n'$, we have done. If $n > n'$, then we have the following inequation;

$$u_n = u_{n'+1} = u_{n'} + 2 \leq \hat{l}_{n'} \leq \hat{l}_n. \square$$

Here, we reprise Theorem 1.7 (2).

Theorem 4.22. *For any positive integer g, there exist two matroids M and N such that $T^{(g)}(M) = T^{(g)}(N)$ and $T^{(\lceil\sqrt{2}g\rceil)}(M) \neq T^{(\lceil\sqrt{2}g\rceil)}(N)$.*

Proof. It is clear from Theorem 4.19, Theorem 4.20, and Proposition 4.21. □

5 Concluding remarks

5.1 Special values of $T^{(2)}$

Theorem 5.1. *We have*

$$\begin{aligned} T(M;x,y) &= T^{(2)}(M;2,2,0,x,2,2,0,y) \\ &= T^{(2)}(M;2,2,x,0,2,2,y,0) \\ &= T^{(2)}(M;2,2,0,x,2,2,y,0). \end{aligned}$$

Proof. We show the first identity $T(M;x,y) = T^{(2)}(M;2,2,0,x,2,2,0,y)$. The other cases can be proved similarly. The right-hand side is written as follows:

$$
\begin{aligned}
&T^{(2)}(M;2,2,0,x,2,2,0,y)\\
&\quad=\sum_{A_1,A_2\subset E}(-1)^{\rho E-\rho(A_1\cap A_2)}(x-1)^{\rho E-\rho(A_1\cup A_2)}(-1)^{|A_1\cap A_2|-\rho(A_1\cap A_2)}(y-1)^{|A_1\cup A_2|-\rho(A_1\cup A_2)}\\
&\quad=\sum_{A_1,A_2\subset E}(-1)^{\rho E-|A_1\cap A_2|}(x-1)^{\rho E-\rho(A_1\cup A_2)}(y-1)^{|A_1\cup A_2|-\rho(A_1\cup A_2)}.
\end{aligned}
$$

We claim that if we fix $A_1\neq\emptyset$ then

$$\sum_{A_1,A_2\subset E}(-1)^{\rho E-|A_1\cap A_2|}(x-1)^{\rho E-\rho(A_1\cup A_2)}(y-1)^{|A_1\cup A_2|-\rho(A_1\cup A_2)}=0.$$

In fact, if $|A_1|$ is odd then for fixed A_2 the terms (A_1,A_2) and $(A_1,A_1\cup A_2)$ have opposite signature since

$$|A_1\cup A_2|\not\equiv|A_1\cup(A_1\cup A_2)|\pmod 2.$$

If $|A_1|$ is even then for fixed A_2

$$\sum_{S\subset A_1}(-1)^{\rho E-|A_1\cap(S\cup A_2)|}(x-1)^{\rho E-\rho(A_1\cup(S\cup A_2))}(y-1)^{|A_1\cup(S\cup A_2)|-\rho(A_1\cup(S\cup A_2))}=0.$$

It is because if S_1 is odd and S_2 is even $(A_1,S_1\cup A_2)$ and $(A_1,S_2\cup A_2)$ have opposite signature since

$$|A_1\cap(S_1\cup A_2)|\not\equiv|A_1\cap(S_2\cup A_2)|\pmod 2.$$

This completes the proof of Theorem 5.1.

□

By Theorem 5.6, we have the following corollary:

Corollary 5.2. *Let M be a vector matroid over $\mathbb{F}_q$, and C_M be the corresponding code. Then*

$$
\begin{aligned}
w_{C_M}(x,y)&=T^{(2)}\left(M;2,2,0,\frac{x+(q-1)y}{x-y},2,2,0,\frac{x}{y}\right)\\
&=T^{(2)}\left(M;2,2,\frac{x+(q-1)y}{x-y},0,2,2,\frac{x}{y},0\right)\\
&=T^{(2)}\left(M;2,2,0,\frac{x+(q-1)y}{x-y},2,2,\frac{x}{y},0\right).
\end{aligned}
$$

Using (6) $\sim$ (8) in [14, p.268], we have the following corollary:

Corollary 5.3. *Let M be a vector matroid. Then*

$$
\begin{aligned}
&\textit{number of basis of } M\\
&\qquad = T^{(2)}(M;2,2,0,1,2,2,0,1)\\
&\qquad = T^{(2)}(M;2,2,1,0,2,2,1,0)\\
&\qquad = T^{(2)}(M;2,2,0,1,2,2,1,0),\\
&\textit{number of independent sets of } M\\
&\qquad = T^{(2)}(M;2,2,0,2,2,2,0,1)\\
&\qquad = T^{(2)}(M;2,2,2,0,2,2,1,0)\\
&\qquad = T^{(2)}(M;2,2,0,2,2,2,1,0),\\
&\textit{number of spanning sets of } M\\
&\qquad = T^{(2)}(M;2,2,0,1,2,2,0,2)\\
&\qquad = T^{(2)}(M;2,2,1,0,2,2,2,0)\\
&\qquad = T^{(2)}(M;2,2,0,1,2,2,2,0).
\end{aligned}
$$

5.2 Witt's problem from the viewpoint of the Tutte polynomials of genus g

In coding theory, it is called Witt's problem that for the Type II codes of length $8n$, to determine the minimum number g that the genus g weight enumerators of those codes are linearly independent. For the case $n = 1$, there is a unique Type II code. Therefore, the crucial problems are the cases $n \geq 2$. The answer is known for the cases $n = 2, 3$, and 4. For $n = 2, 3$, and 4, the answer is $3, 6$, and 10, respectively ([12]).

In this section, we study Witt's problem from the viewpoint of Tutte polynomial of genus g, namely to determine the minimum number g that the genus g Tutte polynomial of those codes are linearly independent. The results are as follows:

Theorem 5.4. *For $n = 2$, the genus one Tutte polynomial $T^{(1)}$ of Type II codes of length $8n$ are linearly independent. For $n = 3$, the genus one Tutte polynomial $T^{(1)}$ of Type II codes of length $8n$ are linearly dependent.*

Proof. We perform brute-force enumeration based on the definition by using MAGMA [2] and MATHEMATICA [15]. The data is given in [5] and [10]. □

Remark 5.5.

- The classification of Type II codes of length $8n$ is known for $n \leq 5$. It is an interesting problem to determine whether the genus g Tutte polynomials of those codes are linearly independent.

- In [4], a relationship between the weight enumerators of codes and the Tutte polynomials of matroids was established. In this remark, we review this relationship to explain the term *genus*.

  Let M be a vector matroid obtained from the $k \times n$ matrix A. Then the row space of A is an $[n, k]$ code over $\mathbb{F}_q$, namely a k-dimensional subspace of of $\mathbb{F}_q^n$. We denote such a code by C_M. The weight enumerator $w_C(x, y)$ of the code C is the homogeneous polynomial

  $$w_C(x, y) = \sum_{c \in C} x^{n-\mathrm{wt}(c)} y^{wt(c)} = \sum_{i=0}^{n} A_i x^{n-i} y^i,$$

  where $A_i = \sharp\{i \mid c = (c_1, \ldots, c_n) \in C, c_i \neq 0\}$. The Tutte polynomial of a vector matroid M and the weight enumerator of C_M have the following relation:

  Theorem 5.6 ([4]). *Let M be a vector matroid on a set $E = \{1, \ldots, n\}$ over $\mathbb{F}_q$. Then*

  $$w_{M_C}(x_1, x_2) = x_2^{n-\dim(M_C)} (x_1 - x_2)^{\dim(M_C)} T\left(M_C; \frac{x_1 + (q-1)x_2}{x_1 - x_2}, \frac{x_1}{x_2}\right).$$

  A generalization of the weight enumerator is known as the weight enumerator of genus g:

  $$w_C^{(g)}(x_a : a \in \mathbb{F}_2^g) = \sum_{v_1, \ldots, v_g \in C} \prod_{a \in \mathbb{F}_2^g} x_a^{n_a(v_1, \ldots, v_g)},$$

  where $n_a(v_1, \ldots, v_g)$ denotes the number of i such that $a = (v_{1i}, \ldots, v_{gi})$. This gives rise to a natural question: is there a generalization of the Tutte polynomial that relates the complete weight enumerator $w_C^{(g)}(x_a : a \in \mathbb{F}_2^g)$? We believe that the Tutte polynomials of genus g is a candidate generalization that answers this. Therefore, we use the term *genus*. So far, we do not have any relations between the complete weight enumerator of genus g and the Tutte polynomials of genus g.

5.3 Future problems

We conclude this paper by stating a few open problems.

1. Following the same policy as in the proof of the main theorem of this paper, we believe that the following conjecture can also be proved.

   Conjecture 5.7. *Let M be a non-separable matroid, let C be a largest circuit of M, and let D be a largest cocircuit of M. Then M can be reconstructed from its Tutte polynomial of genus $|\mathcal{B}(M)|-\max\{|C|,|D|\}+3$.*

2.

   Problem 5.8. Given an arbitrary matroid M, does the upper bound on the genus of the Tutte polynomial of M needed to reconstruct M belong to $o(|B(M)|)$?

3. In 2000, Bollobás, Pebody and Riordan [1] conjectured that almost all graphic matroids are $T^{(1)}$-unique. Related to this conjecture, let us ask the following question:

   Problem 5.9. Are all graphical matroids $T^{(2)}$-unique? More generally, does there exist a positive integer N such that all graphical matroids are $T^{(N)}$-unique or of class $T^{(N)}$?

4. As a direct consequence of Theorem 4.22, we obtain the following proposition:

   Proposition 5.10. *For any positive integer g, there exist two positive numbers c_1, c_2 and two infinite sequences matroids $\{M_i\}$ and $\{N_i\}$ such that $T^{(g)}(M_i) = T^{(g)}(N_i)$ and $T^{(\lceil c_1 g + c_2 \rceil)}(M_i) \neq T^{(\lceil c_1 g + c_2 \rceil)}(N_i)$.*

   In connection with this proposition, we pose the next problem:

   Problem 5.11. Find two infinite sequences of matroids $\{M_i\}$ and $\{N_i\}$ such that the constant c_1 is as small as possible.

   More precisely, we have the following conjecture:

   Conjecture 5.12. *There exist two infinite sequences of matroids $\{M_i\}$ and $\{N_i\}$ such that $T^{(g)}(M_i) = T^{(g)}(N_i)$ and $T^{(g+1)}(M_i) \neq T^{(g+1)}(N_i)$.*

A The difference $T^{(2)}(R_{2n}) - T^{(2)}(Q_{2n})$

Here, we calculate The difference $T^{(2)}(R_{2n}) - T^{(2)}(Q_{2n})$, For a pair (A_1, A_2) of subsets of the ground set E of the matroid M, we define $f(A_1, A_2)$ as

$$\begin{aligned} f(A_1, A_2) :=& (x_1-1)^{\rho(E)-\rho(A_1)}(y_1-1)^{|A_1|-\rho(A_1)}(x_2-1)^{\rho(E)-\rho(A_2)} \\ &\times (y_2-1)^{|A_2|-\rho(A_2)}(x_3-1)^{\rho(E)-\rho(A_1\cap A_2)}(y_3-1)^{|A_1\cap A_2|-\rho(A_1\cap A_2)} \\ &\times (x_4-1)^{\rho(E)-\rho(A_1\cup A_2)}(y_4-1)^{|A_1\cup A_2|-\rho(A_1\cup A_2)}, \end{aligned}$$

For a matroid M on E, we define the following polynomials;

$$h(M; s, t, u) := \sum_{\substack{A_1, A_2 \subset E, \\ |A_1\cap A_2| = s \\ |A_1| = t, \\ |A_2| = u}} f(A_1, A_2), \tag{A.1}$$

From (1.1) and (A.1), it is clear that the following equation holds;

$$T^{(2)}(M; \{x_1, x_2, x_3, x_4\}, \{y_1, y_2, y_3, y_4\}) = \sum_{(s,t,u)\in(E\cup\{0\})^3} h(M; s, t, u).$$

Since both R_{2n} and Q_{2n} are self-dual, we have

$$h(R_{2n}; s, t, u) = h(R_{2n}; t+u-s, 2n-t, 2n-u), \tag{A.2}$$
$$h(Q_{2n}; s, t, u) = h(Q_{2n}; t+u-s, 2n-t, 2n-u). \tag{A.3}$$

Proposition A.1. *If there does not exist n among $s, t, u, t+u-s$, then we have $h(R_{2n}; s, t, u) = h(Q_{2n}; s, t, u)$.*

Proof. If there does not exist n among $s, t, u, t+u-s$, then we have

$$\begin{aligned} f(A_1, A_2) =& (x_1-1)^{\max\{n-t,0\}}(y_1-1)^{\max\{t-n,0\}}(x_2-1)^{\max\{n-u,0\}} \\ &\times (y_2-1)^{\max\{u-n,0\}}(x_3-1)^{\max\{n-s,0\}}(y_3-1)^{max\{s-n,0\}} \\ &\times (x_4-1)^{\max\{n+s-t-u,0\}}(y_4-1)^{\max\{t+u-s-n,0\}} \end{aligned}$$

holds for any pair (A_1, A_2). Therefore, we have

$$\begin{aligned}
&h(R_{2n}; s,t,u)\\
&= h(Q_{2n}; s,t,u)\\
&= \binom{2n}{s}\binom{2n-s}{t-s}\binom{2n-t}{u-s}(x_1-1)^{\max\{n-t,0\}}(y_1-1)^{\max\{t-n,0\}}\\
&\times (x_2-1)^{\max\{n-u,0\}}(y_2-1)^{\max\{u-n,0\}}(x_3-1)^{\max\{n-s,0\}}(y_3-1)^{max\{s-n,0\}}\\
&\times (x_4-1)^{\max\{n+s-t-u,0\}}(y_4-1)^{\max\{t+u-s-n,0\}}.\square
\end{aligned}$$

Proposition A.2. *If the equation $t+u-s=n$ holds, then the equation $h(R_{2n}; s,t,u) = h(Q_{2n}; s,t,u)$ holds.*

Proof. Note that there exist $\binom{2n}{n}$ choice of $A_1 \cup A_2$.
(Case 1) Suppose that $\max\{t,u\} \le n-1$ holds. For R_{2n}, we have

$$f(A_1,A_2) = \begin{cases} (x_1-1)^{n-t}(x_2-1)^{n-u}(x_3-1)^{n-s} & \\ \times(x_4-1)(y_4-1) & \text{if } A_1\cup A_2 \in X_1, X_2 \\ (x_1-1)^{n-t}(x_2-1)^{n-u}(x_3-1)^{n-s} & \text{otherwise.} \end{cases}$$

For Q_{2n}, we have

$$f(A_1,A_2) = \begin{cases} (x_1-1)^{n-t}(x_2-1)^{n-u}(x_3-1)^{n-s} & \\ \times(x_4-1)(y_4-1) & \text{if } A_1\cup A_2 \in X_1, X_3 \\ (x_1-1)^{n-t}(x_2-1)^{n-u}(x_3-1)^{n-s} & \text{otherwise.} \end{cases}$$

For each $A_1 \cup A_2$ of size n, there exist $\binom{n}{t}\binom{t}{u}$ pairs of (A_1, A_2). Therefore,

$$\begin{aligned}
&h(R_{2n}; s,t,u)\\
&= h(Q_{2n}; s,t,u)\\
&= \binom{n}{t}\binom{t}{u}\left(2(x_4-1)(y_4-1) + \binom{2n}{n} - 2\right)\\
&\times (x_1-1)^{n-t}(x_2-1)^{n-u}(x_3-1)^{n-s}.
\end{aligned}$$

(Case 2) Suppose that $\min\{t,u\} < \max\{t,u\} = n$ holds. Without loss of generality, we can assume $t=n$. For R_{2n}, we have

$$f(A_1,A_2) = \begin{cases} (x_1-1)(y_1-1)(x_2-1)^{n-u}(x_3-1)^{n-s} & \\ \times(x_4-1)(y_4-1) & \text{if } A_1\cup A_2 \in \{X_1, X_2\} \\ (x_2-1)^{n-u}(x_3-1)^{n-s} & \text{otherwise.} \end{cases}$$

For Q_{2n}, we have

$$f(A_1, A_2) = \begin{cases} (x_1 - 1)(y_1 - 1)(x_2 - 1)^{n-u}(x_3 - 1)^{n-s} & \\ \times(x_4 - 1)(y_4 - 1) & \text{if } A_1 \cup A_2 \in \{X_1, X_3\} \\ (x_2 - 1)^{n-u}(x_3 - 1)^{n-s} & \text{otherwise.} \end{cases}$$

For each $A_1 = A_1 \cup A_2$ of size n, there exist $\binom{n}{u}$ choices of $A_2 = A_1 \cap A_2$. Therefore, we have

$$\begin{aligned} & h(R_{2n}; s, t, u) \\ &= h(Q_{2n}; s, t, u) \\ &= \binom{n}{u} \left(2(x_1 - 1)(y_1 - 1)(x_4 - 1)(y_4 - 1) + \binom{2n}{n} - 2 \right) \\ &\times (x_2 - 1)^{n-u}(x_3 - 1)^{n-s}. \end{aligned}$$

(Case 3) Let us assume that $t = u = t + u - s = n$ holds. Then, $s = n$ also holds. Therefore, we have

$$\begin{aligned} & h(R_{2n}; s, t, u) \\ &= h(Q_{2n}; s, t, u) \\ &= 2 \prod_{i=1}^{4}(x_i - 1) \prod_{i=1}^{4}(y_i - 1). \square \end{aligned}$$

Proposition A.3. *For any pair of* t *and* u*, the equation* $h(R_{2n}; n, t, u) = h(Q_{2n}; n, t, u)$ *holds.*

Proof. From the equations (A.2), (A.3) and Proposition A.2, we have

$$\begin{aligned} h(R_{2n}; n, t, u) &= h(R_{2n}; t + u - n, 2n - t, 2n - u) \\ &= h(Q_{2n}; t + u - n, 2n - t, 2n - u) \\ &= h(Q_{2n}; n, t, u). \square \end{aligned}$$

Proposition A.4. *For all* (s, t, u) *such that an either* t *and* u *is* n*, and that the equality* $s \neq n$ *and* $t + u - s \neq n$ *hold, we have* $h(R_{2n}; s, t, u) = h(Q_{2n}; s, t, u)$*.*

Proof. Without loss of generality, we can assume $t = n$ and $u \neq n$. At first, we show (A.5), (A.6), and (A.7) hold for each pair (A_1, A_2) such that all of

$$|A_1| = t = n, \quad |A_2| = t \neq n, |A_1 \cap A_2| = s \neq n, \quad |A_1 \cup A_2| = t + u - s \neq n \tag{A.4}$$

hold. From $t \neq n$, we have

$$\rho(A_2) = \min\{t, n\}. \tag{A.5}$$

From $|A_1 \cap A_2| = s \leq n = t = |A_1|$ and the assumption $s \neq n$, we have $|A_1 \cap A_2| \leq n - 1$, i.e., we have

$$\rho(A_1 \cap A_2) = s. \tag{A.6}$$

From $|A_1 \cup A_2| = t + u - s \geq n = t = |A_1|$ and the assumption $t + u - s \neq n$, we have $|A_1 \cup A_2| \geq n + 1$, i.e., we have

$$\rho(A_1 \cup A_2) = n. \tag{A.7}$$

Note that there are $\binom{2n}{n}\binom{n}{s}\binom{n}{u-s}$ pairs of (A_1, A_2) such that (A.4) holds by choosing n elements of A_1 from E at first, choosing s elements of $A_1 \cap A_2$ from A_1 secondly, and choosing $u - s$ elements of $A_2 \setminus A_1$ from $E \setminus A_1$. Among such pairs, $2\binom{n}{s}\binom{n}{u-s}$ pairs satisfy $\rho(A_1) = n - 1$ for both R_{2n} and Q_{2n}. Therefore, we have

$$\begin{aligned}
&h(R_{2n}; s, n, u) \\
&= h(Q_{2n}; s, n, u) \\
&= 2\binom{n}{s}\binom{n}{u-s}(x_1 - 1)(y_1 - 1)(x_2 - 1)^{\max\{0,n-u\}}(y_2 - 1)^{\max\{0,u-n\}} \\
&\times (x_3 - 1)^{\max\{0,n-s\}}(y_3 - 1)^{\max\{0,s-n\}}(y_4 - 1)^{u-s} \\
&+ \left(\binom{2n}{n} - 2\right)\binom{n}{s}\binom{n}{u-s}(x_2 - 1)^{\max\{0,n-u\}}(y_2 - 1)^{\max\{0,u-n\}} \\
&\times (x_3 - 1)^{\max\{0,n-s\}}(y_3 - 1)^{\max\{0,s-n\}}(y_4 - 1)^{u-s}. \square
\end{aligned}$$

Proposition A.5. *For any integer s at least 2, the equality $h(R_{2n}, s, n, n) = h(Q_{2n}, s, n, n)$ holds.*

Proof. From Proposition A.2 and Proposition A.3, we can assume that $|A_1 \cap A_2| = s \leq n-1$ and $|A_1 \cup A_2| = 2n - s \geq n+1$, i.e., $\rho(A_1 \cap A_2) = s$ and $\rho(A_1 \cup A_2) = n$ holds for both cases of R_{2n} and Q_{2n}.

From $X_1 \cap X_2 = \emptyset$ and $X_1 \cap X_3 = \{n\}$, there does not a pair (A_1, A_2) such that $\rho(A_1) = \rho(A_2) = n-1$. Note that there exist $\binom{2n}{n}\binom{n}{s}\binom{n}{n-s}$ pairs of (A_1, A_2) such that $|A_1| = |A_2| = n$ and that $|A_1 \cap A_2| = s$. Among those pairs, $\rho(A_1) = \rho(A_2) + 1 = n$ holds if and only if

$$\begin{cases} A_2 \in \{X_1, X_2\} & \text{for the case } R_6, \\ A_2 \in \{X_1, X_3\} & \text{for the case } Q_6. \end{cases}$$

The number of pairs of (A_1, A_2) with $\rho(A_1) = \rho(A_2)+1 = n$ is $2\binom{n}{s}\binom{n}{n-s}$ for both cases of R_{2n} and Q_{2n}. There are also $2\binom{n}{s}\binom{n}{n-s}$ pairs of (A_1, A_2) such that $\rho(A_2) = \rho(A_1) + 1 = n$ for both cases of R_{2n} and Q_{2n}. The other $\left(\binom{2n}{s} - 4\right)\binom{2n-s}{n-s}\binom{n}{n-s}$ pairs of (A_1, A_2) satisfies $\rho(A_1) = \rho(A_2) = n$. Therefore, for $s \geq 2$, we have

$$\begin{aligned} & h(R_{2n}, s, n, n) \\ &= h(Q_{2n}, s, n, n) \\ &= 2\binom{n}{s}\binom{n}{n-s}(x_1 - 1)(y_1 - 1)(x_3 - 1)^{n-s}(y_4 - 1)^{n-s} \\ &+ 2\binom{n}{s}\binom{n}{n-s}(x_2 - 1)(y_2 - 1)(x_3 - 1)^{n-s}(y_4 - 1)^{n-s} \\ &+ \left(\binom{2n}{s} - 4\right)\binom{2n-s}{n-s}\binom{n}{n-s}(x_3 - 1)^{n-s}(y_4 - 1)^{n-s}. \square \end{aligned}$$

From Propositions A.1, A.2, A.3, A.4, and A.5,

$$\begin{aligned} & T^{(2)}(R_{2n}, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) - T^{(2)}(Q_{2n}, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \\ &= h(R_{2n}; 0, n, n) + h(R_{2n}; 1, n, n) - h(Q_{2n}; 0, n, n) - h(Q_{2n}; 1, n, n) \quad \text{(A.8)} \end{aligned}$$

holds for each integer n at least 3.

From here, we calculate $h(R_{2n}; 0, n, n)$, $h(R_{2n}; 1, n, n)$, $h(Q_{2n}; 0, n, n)$, and $h(Q_{2n}; 1, n, n)$.

Note that there exist $\binom{2n}{n}\binom{n}{1}\binom{n}{1} = n^2\binom{2n}{n}$ pairs of (A_1, A_2) such that $|A_1| = |A_2| = n$ and $|A_1 \cap A_2| = 1$ hold by choosing n elements of A_1 from E first, and choosing a unique element of $A_1 \cap A_2$ from A_1 secondly, and choosing a unique element of $E \setminus (A_1 \cup A_2)$ from $E \setminus A_1$ lastly for both cases of R_{2n} and Q_{2n}.

In the matroid R_{2n}, there does not exist a pair (A_1, A_2) such that $\rho(A_1) = \rho(A_2) = n$ as same as the condition $s \geq 2$. Therefore, we have

$$\begin{aligned} &h(R_{2n}, 1, n, n) \\ &= 2n^2\left((x_1-1)(y_1-1) + (x_2-1)(y_2-1)\right)(x_3-1)^{n-1}(y_4-1)^{n-1} \\ &+ \left(\binom{2n}{n} - 4\right) n^2 (x_3-1)^{n-1}(y_4-1)^{n-1}. \end{aligned} \tag{A.9}$$

In the matroid Q_{2n}, there exist two pairs of (A_1, A_2) such that $\rho(A_1) = \rho(A_2) = |A_1| - 1 = |A_2| - 1 = n - 1$ and $|A_1 \cap A_2| = 1$, that is the case of $\{A_1, A_2\} = \{X_1, X_3\}$. We count the number of (A_1, A_2) such that $\rho(A_1) = \rho(A_2) - 1 = |A_1| - 1 = |A_2| - 1 = n - 1$ and $|A_1 \cap A_2| = 1$. Let us assume $A_1 = X_1$. Then, there exist $\binom{n}{1}\binom{n}{1} = n^2$ sets such that both $|A_2| = n$ and $|A_1 \cap A_2| = 1$ hold. Among these n^2 choices of A_2, $n^2 - 1$ sets of A_2 other than the case $A_2 = X_3$ satisfies $\rho(A_2) = n$. There are $n^2 - 1$ choices of A_2 such that $A_1 = X_3$. There exist $2(n^2-1)$ pairs of (A_1, A_2) such that $\rho(A_1) = \rho(A_2) - 1 = |A_1| - 1 = |A_2| - 1 = n - 1$ and $|A_1 \cap A_2| = 1$. There also exist $2(n^2-1)$ pairs of (A_1, A_2) such that $\rho(A_1) - 1 = \rho(A_2) = |A_1| - 1 = |A_2| - 1 = n - 1$ and $|A_1 \cap A_2| = 1$. The other $n^2\binom{2n}{n} - 2 - 4(n^2-1)$ pairs of (A_1, A_2) satisfy $\rho(A_1) = \rho(A_2) = |A_1| = |A_2| = n$. Therefore, we have

$$\begin{aligned} &h(Q_{2n}; 1, n, n) \\ &= 2(x_1-1)(y_1-1)(x_2-1)(y_2-1)(x_3-1)^{n-1}(y_4-1)^{n-1} \\ &+ 2(n^2-1)(x_1-1)(y_1-1)(x_3-1)^{n-1}(y_4-1)^{n-1} \\ &+ 2(n^2-1)(x_2-1)(y_2-1)(x_3-1)^{n-1}(y_4-1)^{n-1} \\ &+ \left(\binom{2n}{n}n^2 - 4n^2 + 2\right)(x_3-1)^{n-1}(y_4-1)^{n-1}. \end{aligned} \tag{A.10}$$

There are $\binom{2n}{n}$ pairs of (A_1, A_2) such that $|A_1 \cap A_2| = 0$, $|A_1| = |A_2| = n$ for both cases of R_{2n} and Q_{2n}.

In the matroid R_{2n}, there exist two pairs of (A_1, A_2) such that $\rho(A_1) = \rho(A_2) = |A_1| - 1 = |A_2| - 1 = n - 1$ and $A_1 \cap A_2 = \emptyset$, that is the case of $\{A_1, A_2\} = \{X_1, X_2\}$.

The condition $A_1 \notin \{X_1, X_2\}$ and $A_2 = \bar{A}_1$ yields $A_2 \notin \{X_1, X_2\}$. Then, $\binom{2n}{n} - 2$ pairs of (A_1, A_2) with $A_1 \cap A_2 = \emptyset$ satisfy $\rho(A_1) = \rho(A_2) = |A_1| = |A_2| = n$. Therefore, we have

$$\begin{aligned} &h(R_{2n}; 0, n, n) \\ &= \left(2(x_1-1)(y_1-1)(x_2-1)(y_2-1) + \binom{2n}{n} - 2\right)(x_3-1)^n(y_4-1)^n. \end{aligned} \tag{A.11}$$

In the matroid Q_{2n}, there exist two pairs of (A_1, A_2) such that $\rho(A_1) = \rho(A_2) - 1 = |A_1| - 1 = |A_2| - 1 = n - 1$ and $A_1 \cap A_2| = \emptyset$, that is the case of $A_1 \in \{X_1, X_3\}$. There also exist two pairs of (A_1, A_2) such that $\rho(A_1) - 1 = \rho(A_2) = |A_1| - 1 = |A_2| - 1 = n - 1$ and $A_1 \cap A_2 = \emptyset$. The other $\binom{2n}{n} - 4$ pairs of (A_1, A_2) with $A_1 \cap A_2 = \emptyset$ satisfy $\rho(A_1) = \rho(A_2) = |A_1| = |A_2| = n$. Therefore, we have

$$\begin{aligned} &h(Q_{2n}; 0, n, n) \\ &= \left(2(x_1-1)(y_1-1) + 2(x_2-1)(y_2-1) + \binom{2n}{n} - 4\right)(x_3-1)^n(y_4-1)^n. \end{aligned} \tag{A.12}$$

From (A.8), (A.9), (A.10), (A.11), and (A.12), we have

$$\begin{aligned} &T^{(2)}(R_{2n}; x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) - T^{(2)}(Q_{2n}; x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \\ &= 2(x_1y_1 - x_1 - y_1)(x_2y_2 - x_2 - y_2)(x_3y_4 - x_3 - y_4)(x_3-1)^{n-1}(y_4-1)^{n-1}. \end{aligned}$$

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