THE JOINT WEIGHT ENUMERATORS AND SIEGEL MODULAR FORMS

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ABSTRACT. The weight enumerator of a binary doubly even self dual code is an isobaric polynomial in the two generators of the ring of invariants of a certain group of order 192. The aim of this note is to study the ring of coefficients of that polynomial, both for standard and joint weight enumerators.

1. Introduction

It is well known(see [7]) that any modular form whose Fourier coefficients lie in $\mathbf{Z}$ can be written as a polynomial over $\mathbf{Z}$ in $E_4(\tau), E_6(\tau)$ and $\Delta(\tau) = \frac{1}{2^6 \cdot 3^3}(E_4^3 - E_6^2)$, where E_k is the normalized Eisenstein series of weight k. Furthermore, Igusa[6] and Nagaoka[8] determined the minimal set of generators over $\mathbf{Z}$ of the graded ring of Siegel modular forms of degree 2 whose Fourier coefficients lie in $\mathbf{Z}$ and of the graded ring of symmetric Hilbert modular forms for the real quadratic fields $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{5})$, respectively.

On the other hand, recently the connections between the coding theory and theory of various modular forms have been well studied(see [1], [3]). For instance, it is well known that the weight enumerator of every doubly even self dual binary code is a polynomial in two generators, the complete weight enumerator $W_{e_8}^{(1)}$ of the Hamming code and the complete weight enumerator $W_{g_{24}}^{(1)}$ of the Golay code. The graded ring $\mathbf{C}[W_C^{(1)}]$ of the weight enumerators of all doubly even self dual binary codes is isomorphic to the graded ring $\mathbf{C}[E_4, \Delta]$ of elliptic modular forms[3]; explicitly,

$$\mathbf{C}[W_C^{(1)}] = \mathbf{C}[W_{e_8}^{(1)}, W_{g_{24}}^{(1)}] \cong \mathbf{C}[E_4, \Delta]. \tag{1.1}$$

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Since the coefficients of any weight enumerators of codes are in $\mathbf{Z}$, a natural question is if any weight enumerator of all doubly even self dual binary codes can be written as a polynomial in $W_{e_8}^{(1)}$ and $W_{g_{24}}^{(1)}$ over a smaller ring than $\mathbf{C}$. It turns out that we can replace $\mathbf{C}$ in the equality in the equation (1.1) by the smaller ring $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}]$. The main result of this paper is that we extend this problem to genus 2 case as well.

We start with the definitions and the known facts which are needed in this note. Let $\mathfrak{S}_n$ be the Siegel upper-half space of degree n and denote by $A(\Gamma_n)_k$ the ring of modular forms of weight k on $\Gamma_n = Sp_{2n}(\mathbf{Z})$ over $\mathbf{C}$. If f is in $A(\Gamma_n)_k$, then $f(\tau)$ can be expanded into a Fourier series of the following form:

$$f(\tau) = \sum_{s \geqq 0} a_f(s) \exp(2\pi\sqrt{-1}\mathrm{trace}(s\tau)) = \sum_{s_{ii} \geqq 0} \left(\sum a_f(s) \cdot \prod_{i<j} q_{ij}^{2s_{ij}} \right) \prod_{i=1}^{n} q_{ii}^{s_{ii}},$$

where $q_{ij} = \exp(2\pi\sqrt{-1}\tau_{ij})$ and s runs over the set of half-integral positive (semi-definite) matrices of degree n. For any subring R of $\mathbf{C}$ we denote by $A_R(\Gamma_n)_k$ the R-module consisting of those $f \in A(\Gamma_n)_k$ such that $a_f(s)$ is in R for every s and by $A_R(\Gamma_n) := \oplus_{k \geq 0} A_R(\Gamma_n)_k$ taken in $A(\Gamma_n) := \oplus_{k\geq 0} A(\Gamma_n)_k$; then $A_R(\Gamma_n)$ forms a graded integral ring over R. The explicit structure of the ring $A_{\mathbf{Z}}(\Gamma_n)$ is known only for $n = 1, 2$ and we shall use them later.

Let $\mathfrak{m} = (m'\ m'')$, with $m', m'' \in \mathbf{F}_2^n$; then the theta constants with characteristic $\mathfrak{m}$ is defined as

$$\theta_{\mathfrak{m}}(\tau) = \sum_{p \in \mathbf{Z}^n} \exp 2\pi\sqrt{-1} \left\{ \frac{1}{2}(p + \frac{1}{2}m')\tau\ {}^t(p + \frac{1}{2}m') + \frac{1}{2}(p + \frac{1}{2}m')m'' \right\}.$$

Let C be a (linear) code of length k over $\mathbf{F}_2$. The weight enumerator $W_C^{(n)}(x_a : a \in \mathbf{F}_2^n)$ of degree n is defined as

$$W_C^{(n)} = W_C^{(n)}(x_a : a \in \mathbf{F}_2^n) = \sum_{v_1,\ldots,v_k \in C} \prod_{a \in \mathbf{F}_2^n} x_a^{n_a(v_1,\ldots,v_k)},$$

where $n_a(v_1, \ldots, v_k)$ denotes the number of i such that $a = (v_{1i}, \ldots, v_{ki})$. We note that $W_C^{(n)}$ is a homogeneous polynomial of degree k with non-negative integers as its coefficients. For any subring R of $\mathbf{C}$ we denote by $R[W_C^{(n)}]$ the graded ring generated by the weight enumerators of degree n of all doubly even self dual codes of any length over R. It is known that the Broué-Enguehard map

$Th : x_a \mapsto \theta_{a0}(2\tau), a \in \mathbf{F}_2^n$, gives the $\mathbf{C}$–algebra homomorphism from $\mathbf{C}[W_C^{(n)}]$ to $A(\Gamma_n)^{(4)} = \oplus_{k \geqq 0, k \equiv 0 \pmod 4} A(\Gamma_n)_k$. In particular, it gives the isomorphisms $\mathbf{C}[W_C^{(n)}] \cong A(\Gamma_n)^{(4)}$ when $n = 1, 2$(see [10]). In the next section we explain our problem dealing with the case when $n = 1$. The main theme of this note is to investigate this in the case when $n = 2$.

2. The case when $n = 1$

In this section, we discuss the case when $n = 1$ (and may omit $n = 1$ in the notation of the weight enumerator for the sake of simplicity). Before proving the assertion, we modify our setting. We started from the fact(see [3]) , called *Gleason Theorem*, that $\mathbf{C}[W_C]$ is generated by W_{e_8} and $W_{g_{24}}$ over $\mathbf{C}$, where

$$W_{e_8} = x_0^8 + 14x_0^4x_1^4 + x_1^8,$$
$$W_{g_{24}} = x_0^{24} + 759x_0^{16}x_1^8 + 2576x_0^{12}x_1^{12} + 759x_0^8x_1^{16} + x_1^{24}.$$

The doubly even self dual code of length 8 is unique (up to isomorphism), however, we may take another doubly even self dual code of length 24 instead of g_{24}. There exist 7 indecomposable doubly even self dual codes of length 24(see [11]):

$$d_{12}^2, d_{10}e_7^2, d_8^3, d_6^4, d_{24}, d_4^6, g_{24}.$$

We call them $C_{24,1}, \ldots, C_{24,7}$. The following table gives the values a_i, b_i, if we write

$$W_{C_{24,i}} = a_i W_{e_8}^3 + b_i W_{g_{24}}, i = 1, 2, \cdots, 7.$$

	$C_{24,1}$	$C_{24,2}$	$C_{24,3}$	$C_{24,4}$	$C_{24,5}$	$C_{24,6}$	$C_{24,7}$
a_i	5/7	4/7	3/7	2/7	11/7	1/7	0
b_i	2/7	3/7	4/7	5/7	−4/7	6/7	1

We state the following proposition.

Proposition 2.1. *Let $\mathcal{R}$ be a ring such that $\mathbf{Z} \subseteq \mathcal{R} \subseteq \mathbf{C}$. Then we have*

$$\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{C_{24,4}}] \textit{ if and only if } \mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}] \subseteq \mathcal{R},$$

$$\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{C_{24,7}}] \textit{ if and only if } \mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}] \subseteq \mathcal{R},$$

and for $i = 1, 2, 3, 5, 6$,

$$\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{C_{24,i}}] \textit{ if and only if } \mathbf{Z}[\frac{1}{2}, \frac{1}{3}] \subseteq \mathcal{R}.$$

Before proceeding to the proof, we recall the modular forms for Γ_1 over $\mathbf{Z}$. If we denote by E_k the Eisenstein series of even weight k normalized as $E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$, $q = e^{2\pi i\tau}$, and if we put $\Delta = 2^{-6}3^{-3}(E_4^3 - E_6^2)$, then it is well known(see [7]) that $A_{\mathbf{Z}}(\Gamma_1) = \mathbf{Z}[E_4, E_6, \Delta]$. Moreover, we have $A_{\mathbf{Z}}(\Gamma_1)^{(4)} = \mathbf{Z}[E_4, \Delta]$.

Proof of **Proposition 2.1** First we consider the case when $C_{24,7} \cong g_{24}$. Suppose that we have the equality $\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{g_{24}}]$. We pick the doubly even self dual code $C_{32,50}$ of length 32, which is No.50 in the list taken from Sloane's homepage(http://www.research.att.com/ njas/). Direct computation gives

$$W_{C_{32,50}} = \frac{1}{42}W_{e_8}^4 + \frac{41}{42}W_{e_8}W_{g_{24}}.$$

Therefore $\mathcal{R}$ must contain $\frac{1}{2}, \frac{1}{3}, \frac{1}{7}$.

Conversely, suppose that $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}] \subseteq \mathcal{R}$. The inclusion $\mathcal{R}[W_C] \supseteq \mathcal{R}[W_{e_8}, W_{g_{24}}]$ is trivial and we show the converse. Let C be a doubly even self dual code of length k. Then $Th(W_C)$ is in $A(\Gamma_1)_{\frac{k}{2}}$ and $Th(W_C)$ can be expressed in the form

$$Th(W_C) = W_C(\theta_{00}(2\tau), \theta_{10}(2\tau)) = \sum c_{ab}E_4^a\Delta^b, \text{ for some } \ c_{ab} \in \mathbf{Z}.$$

Since

$$E_4(\tau) = Th(W_{e_8}), \qquad \Delta(\tau) = \frac{1}{2^5 3 \cdot 7}\left(Th(W_{e_8})^3 - Th(W_{g_{24}})\right),$$

we get

$$\begin{aligned} Th(W_C) &= \sum c_{ab}E_4^a\Delta^b \\ &= \sum c_{ab}Th(W_{e_8})^a\left\{\frac{1}{2^5 3 \cdot 7}\left(Th(W_{e_8})^3 - Th(W_{g_{24}}))\right)\right\}^b \\ &= \sum \widetilde{c_{a'b'}}Th(W_{e_8})^{a'}Th(W_{g_{24}})^{b'}, \end{aligned}$$

in which $\widetilde{c_{a'b'}}$'s are elements of $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}]$. Therefore W_C is contained in $\mathcal{R}[W_{e_8}, W_{g_{24}}]$. This completes the proof of the case when $C_{24,7} \cong g_{24}$.

For other cases in the proposition, a similar method can be applied and so we omit the detailed proof. □

3. The case when $n = 2$

In this section, we shall discuss the case when $n = 2$ (and may omit $n = 2$ in the notation of the weight enumerator). Our starting point is the following equality given in [4]:

$$\mathbf{C}[W_C] = \mathbf{C}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}],$$

where

$$\begin{aligned} W_{g_{24}} = {} & (24) + 759(16,8) + 2576(12,12) + 212520(12,4,4,4) \\ & + 340032(10,6,6,2) + 22770(8,8,8) + 1275120(8,8,4,4) \\ & + 4080384(6,6,6,6), \end{aligned}$$

$$W_{d_k^+} = \frac{1}{2^2} \sum_{\beta,\gamma \in \mathbf{F}_2^2} \left(\sum_{\alpha \in \mathbf{F}_2^2} (-1)^{\alpha\cdot\beta} x_{\alpha+\gamma} x_\alpha \right)^{\frac{k}{2}}, \; k = 8, 24, 32, 40$$

with the usual inner product $\cdot$ of $\mathbf{F}_2^2$. Here we write e_8 instead of d_8^+ and use the convention $(*, *, \ldots)$ to express the symmetric polynomials, such as $(24) = x_{00}^{24} + x_{01}^{24} + x_{10}^{24} + x_{11}^{24}$, etc. In [4] it was shown that $W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{40}^+}$ are algebraically independent over $\mathbf{C}$ and there exists a unique relation, which is explicitly given in [12]:

$$\begin{aligned} W_{d_{32}^+}^2 = {} & -113 \cdot 32621 \cdot 3^{-4}5^{-1}7^{-2}41^{-1}W_{e_8}^8 - 2^8 60289 \cdot 3^{-4}5^{-1}7^{-2}11^{-1}41^{-1}W_{e_8}^5 W_{g_{24}} \\ & + 2^4 821477 \cdot 3^{-4}5^{-1}7^{-1}11^{-1}41^{-1}W_{e_8}^5 W_{d_{24}^+} + 2 \cdot 751 \cdot 3^{-2}7^{-1}41^{-1}W_{e_8}^4 W_{d_{32}^+} \\ & - 2^9 11^2 \cdot 3^{-3}5^{-1}7^{-1}41^{-1}W_{e_8}^3 W_{d_{40}^+} + 2^{14}163 \cdot 3^{-4}7^{-2}11^{-2}41^{-1}W_{e_8}^2 W_{g_{24}}^2 \\ & + 2^{11}73 \cdot 79 \cdot 3^{-4}7^{-1}11^{-2}41^{-1}W_{e_8}^2 W_{g_{24}} W_{d_{24}^+} - 2^6 107 \cdot 499 \cdot 3^{-4}11^{-2}41^{-1}W_{e_8}^2 W_{d_{24}^+}^2 \\ & - 2^8 389 \cdot 3^{-2}7^{-1}11^{-1}41^{-1}W_{e_8} W_{g_{24}} W_{d_{32}^+} + 2^4 5 \cdot 197 \cdot 3^{-2}11^{-1}41^{-1}W_{e_8} W_{d_{24}^+} W_{d_{32}^+} \\ & + 2^{12}3^{-1}5^{-1}7^{-1}41^{-1}W_{g_{24}} W_{d_{40}^+} + 2^9 3^{-1}5^{-1}41^{-1}W_{d_{24}^+} W_{d_{40}^+}. \end{aligned}$$

So, finally we state our main result:

Theorem 3.1. *Let $\mathcal{R}$ be a ring such that $\mathbf{Z} \subseteq \mathcal{R} \subseteq \mathbf{C}$. Then we have*

$$\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}]$$

if and only if $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}] \subseteq \mathcal{R}$.

The proof of this theorem is carried out by a similar method to that of Proposition 2.1. We recall that $A(\Gamma_2)$ is generated by homogeneous elements $\psi_4, \psi_6, \chi_{10}, \chi_{12}, \chi_{35}$ over $\mathbf{C}$, each with the subscript as its weight. The normalization is made as follows(we follow the notation in [6]):

$$\begin{aligned}
\psi_4(\tau) &= 1 + \cdots ,\\
\psi_6(\tau) &= 1 + \cdots ,\\
\chi_{10}(\tau) &= (q_{11}q_{22} + \cdots)(\pi\tau_{12})^2 + \cdots ,\\
\chi_{12}(\tau) &= (q_{11}q_{22} + \cdots) + \cdots ,\\
\chi_{35}(\tau) &= (q_{11}^2 q_{22}^2 (q_{11} - q_{22}) + \cdots)(\pi\tau_{12}) + \cdots .
\end{aligned}$$

We put

$$X_4 = \psi_4, \quad X_6 = \psi_6, \quad X_{10} = -2^2\chi_{10}, \quad X_{12} = 2^2 3\chi_{12}, \quad X_{35} = 2^2 i\chi_{35},$$

and

$$Y_{12} = 2^{-6}3^{-3}(X_4^3 - X_6^2) + 2^4 3^2 X_{12}, X_{16} = 2^{-2}3^{-1}(X_4X_{12} - X_6X_{10}),$$

$$X_{18} = 2^{-2}3^{-1}(X_6X_{12} - X_4^2X_{10}), X_{24} = 2^{-3}3^{-1}(X_{12}^2 - X_4X_{10}^2),$$

$$X_{28} = 2^{-1}3^{-1}(X_4X_{24} - X_{10}X_{18}), X_{30} = 2^{-1}3^{-1}(X_6X_{24} - X_4X_{10}X_{16}),$$

$$X_{36} = 2^{-1}3^{-2}(X_{12}X_{24} - X_{10}^2X_{16}), X_{40} = 2^{-2}(X_4X_{36} - X_{10}X_{30}),$$

$$X_{42} = 2^{-2}3^{-1}(X_{12}X_{30} - X_4X_{10}X_{28}), X_{48} = 2^{-2}(X_{12}X_{36} - X_{24}^2).$$

Igusa [6] showed that *the fifteen elements*

$$X_4, X_6, X_{10}, X_{12}, Y_{12}, X_{16}, X_{18}, X_{24}, X_{28}, X_{30}, X_{35}, X_{36}, X_{40}, X_{42}, X_{48}$$

form a minimal set of generators of $A_{\mathbf{Z}}(\Gamma_2)$ *over* $\mathbf{Z}$. For our purpose, we deduce the following lemma.

Lemma 3.2. *The ring $A_{\mathbf{Z}}(\Gamma_2)^{(4)}$ can be generated over $\mathbf{Z}$ by the following thirty elements:*

$$X_4, X_{12}, Y_{12}, X_{16}, X_{24}, X_{28}, X_{36}, X_{40}, X_{48},$$

and

$$\begin{array}{cccccc}
X_6^2\ , & X_6X_{10}, & X_6X_{18}\ , & X_6X_{30}\ , & X_6X_{42}\ , & X_6X_{35}^2, \\
 & X_{10}^2\ , & X_{10}X_{18}\ , & X_{10}X_{30}\ , & X_{10}X_{42}\ , & X_{10}X_{35}^2, \\
 & & X_{18}^2\ , & X_{18}X_{30}\ , & X_{18}X_{42}\ , & X_{18}X_{35}^2 \\
 & & & X_{30}^2, & X_{30}X_{42}, & X_{30}X_{35}^2, \\
 & & & & X_{42}^2, & X_{42}X_{35}^2, \\
 & & & & & X_{35}^4.
\end{array}$$

Proof. This is derived from the usual argument on the graded ring. See Chapter III in [5]. □

We notice that the thirty elements in Lemma 3.2 do *not* form a minimal set of generators of $A_{\mathbf{Z}}(\Gamma_2)^{(4)}$, however, it is enough for our purpose. We put

$$\mathcal{Z} = \mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}][Th(W_{e_8}), Th(W_{g_{24}}), Th(W_{d_{24}^+}), Th(W_{d_{32}^+}), Th(W_{d_{40}^+})].$$

By the following two lemmas, we show that the thirty elements in Lemma 3.2 are in $\mathcal{Z}$.

Lemma 3.3. *If the elements $X_4, X_{12}, X_6^2, X_6X_{10}, X_{10}^2$ are in $\mathcal{Z}$, then the remaining twenty five elements in Lemma 3.2 are also in $\mathcal{Z}$.*

Proof. This is derived from the definition of each element and the formula

$$\begin{aligned}
X_{35}^2 &= (-2^2X_4^2X_{16} + Y_{12}^2)X_{36}X_{10} + (-2^6X_4X_{12}^3 + 2^3Y_{12}X_{28})X_{10}^3 \\
&\quad + (2Y_{12}X_{18} + 2^{10}X_{30})X_{10}^4 + 3\cdot 61X_4^2X_{12}X_{10}^5 - 2\cdot 73X_4X_6X_{10}^6 + 2^{10}5^5X_{10}^7,
\end{aligned}$$

which was given in [6]. For example, by the assumption that X_4, X_6^2, X_{12} are in $\mathcal{Z}$, we conclude that $Y_{12} = 2^{-6}3^{-3}(X_4^3 - X_6^2) + 2^43^2X_{12}$ is in $\mathcal{Z}$. Since the assertion can be checked directly, we omit the detailed proof. □

Lemma 3.4. *The elements $X_4, X_{12}, X_6^2, X_6X_{10}, X_{10}^2$ are in $\mathcal{Z}$.*

Proof. It is known that the Broué-Enguehard map gives an isomorphism $\mathbf{C}[W_{e_8}, h_{12}, F_{20}, W_{g_{24}}, W_{d_{40}^+}] \cong A(\Gamma_2)^{(2)}$, where

$$\begin{aligned}
h_{12} &= (12) - 33(8,4) + 330(4,4,4) + 792(6,2,2,2),\\
F_{20} &= (20) - 19(16,4) - 336(14,2,2,2) - 494(12,8) + 716(12,4,4)\\
&\quad + 1038(8,8,4) + 7632(10,6,2,2) + 106848(6,6,6,2) + 129012(8,4,4,4).
\end{aligned}$$

The relations among the polynomials and Siegel modular forms can be given explicitly as follows (*cf.* [9]);

$$\begin{aligned}
W_{d_{24}^+} &= 11^2 3^{-2} 7^{-1} W_{e_8}^3 + 2 \cdot 3^{-2} h_{12}^2 - 2^3 7^{-1} W_{g_{24}},\\
W_{d_{32}^+} &= 43 \cdot 53 \cdot 3^{-4} 7^{-1} W_{e_8}^4 + 2^4 5 \cdot 23 \cdot 3^{-5} 11^{-1} W_{e_8} h_{12}^2\\
&\quad - 2^6 43 \cdot 3^{-2} 7^{-1} 11^{-1} W_{e_8} W_{g_{24}} + 2^6 3^{-5} h_{12} F_{20},\\
W_{d_{40}^+} &= 3 \cdot 19 \cdot 7^{-1} W_{e_8}^5 + 2 \cdot 5 \cdot 7 \cdot 557 \cdot 3^{-7} 11^{-1} W_{e_8}^2 h_{12}^2\\
&\quad - 2^3 5 \cdot 19 \cdot 7^{-1} 11^{-1} W_{e_8}^2 W_{g_{24}} + 2^6 5^2 3^{-7} W_{e_8} h_{12} F_{20} + 2^2 5 \cdot 41 \cdot 3^{-7} F_{20}^2,
\end{aligned}$$

and

$$Th(W_{e_8}) = \psi_4, Th(h_{12}) = \psi_6, Th(F_{20}) = \psi_4\psi_6 + 2^{12}3^4\chi_{10},$$

$$Th(W_{g_{24}}) = 11 \cdot 2^{-1}3^{-2}\psi_4^3 + 7 \cdot 2^{-1}3^{-2}\psi_6^2 - 2^{10}3^2 7 \cdot 11\chi_{12}.$$

So, we have

$$\begin{aligned}
X_4 &= Th(W_{e_8}),\\
X_{12} &= Th(-2^{-10}3^{-1}7^{-1}W_{e_8}^3 + 2^{-8}3^{-1}7^{-1}11^{-1}W_{g_{24}} + 2^{-10}3^{-1}11^{-1}W_{d_{24}^+}),\\
X_6^2 &= Th\left(-11^2 2^{-1}7^{-1}W_{e_8}^3 + 2^2 3^2 7^{-1} W_{g_{24}} + 3^2 2^{-1} W_{d_{24}^+}\right),\\
X_6X_{10} &= Th(-5 \cdot 53 \cdot 2^{-16}3^{-1}7^{-1}W_{e_8}^4 + 5 \cdot 2^{-9}3^{-1}7^{-1}11^{-1}W_{e_8}W_{g_{24}}\\
&\quad + 53 \cdot 2^{-13}3^{-1}11^{-1}W_{e_8}W_{d_{24}^+} - 3 \cdot 2^{-16}W_{d_{32}^+}),\\
X_{10}^2 &= Th(-461 \cdot 2^{-25}3^{-1}5^{-1}7^{-1}41^{-1}W_{e_8}^5 + 2^{-18}3^{-1}7^{-1}11^{-1}41^{-1}W_{e_8}^2 W_{g_{24}}\\
&\quad + 13 \cdot 2^{-21}3^{-1}11^{-1}41^{-1}W_{e_8}^2 W_{d_{24}^+} - 3 \cdot 2^{-25}41^{-1}W_{e_8}W_{d_{32}^+}\\
&\quad + 2^{-22}3^{-1}5^{-1}41^{-1}W_{d_{40}^+}).
\end{aligned}$$

This shows Lemma 3.4. □

Proof of **Theorem 3.1**. Suppose that $\mathcal{R}[W_C] = \mathcal{R}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}]$. Since the weight enumerator $W_{d_{48}^+}$ is uniquely expressed with our fixed generators as

$$\begin{aligned}
W_{d_{48}^+} &= 23 \cdot 22229 \cdot 2^{-2}3^{-2}5^{-1}7^{-2}41^{-1}W_{e_8}^6 - 2^5 13 \cdot 23 \cdot 3^{-2}7^{-2}11^{-1}41^{-1}W_{e_8}^3 W_{g_{24}} \\
&\quad + 2 \cdot 23 \cdot 113 \cdot 3^{-2}7^{-1}11^{-1}41^{-1}W_{e_8}^3 W_{d_{24}^+} - 3^2 5 \cdot 23 \cdot 2^{-2}41^{-1}W_{e_8}^2 W_{d_{32}^+} \\
&\quad + 2^4 7 \cdot 23 \cdot 3^{-1}5^{-1}41^{-1}W_{e_8} W_{d_{40}^+} - 2^9 19 \cdot 3^{-2}7^{-2}11^{-2}W_{g_{24}}^2 \\
&\quad + 2^6 23 \cdot 3^{-2}7^{-1}11^{-2}W_{g_{24}} W_{d_{24}^+} + 2 \cdot 23 \cdot 37 \cdot 3^{-2}11^{-2}W_{d_{24}^+}^2,
\end{aligned}$$

we see that $\mathcal{R}$ must contain $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}]$.

Conversely, suppose that $\mathcal{R}$ contains $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}]$. It is enough to show that W_C is in $\mathcal{R}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}]$ for *any* doubly even self dual code C. Take any doubly even self dual code C of length k. Then $Th(W_C)$ is in $A_{\mathbf{Z}}(\Gamma_2)_k$, with weight $\frac{k}{2}$ and $k \equiv 0 \pmod 8$. By Lemma 3.2, $Th(W_C)$ is expressed as the polynomial of the thirty elements $X_4, \ldots, X_6^2, \ldots$ over $\mathbf{Z}$, say

$$Th(W_C) = \sum_{a,\cdots,b,\cdots} c_{a\cdots b\cdots} X_4^a \cdots X_6^{2b} \cdots ,$$

in which $c_{a\cdots b\cdots}$'s are integers. By Lemmas 3.3 and 3.4, all thirty elements are in $\mathcal{Z}$ and we have

$$Th(W_C) = Th(\sum_{a',b',c',d',e' \in \mathbf{Z}} \widetilde{c_{a'b'c'd'e'}} W_{e_8}^{a'} W_{g_{24}}^{b'} W_{d_{24}^+}^{c'} W_{d_{32}^+}^{d'} W_{d_{40}^+}^{e'}),$$

in which the coefficients $\widetilde{c_{a'b'c'd'e'}}$ are in $\mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{41}]$. At any rate, W_C is in $\mathcal{R}[W_{e_8}, W_{g_{24}}, W_{d_{24}^+}, W_{d_{32}^+}, W_{d_{40}^+}]$. This completes the proof of Theorem 3.1. $\square$

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References

[1] Bannai, E., Modular invariance property of association schemes, type II codes over finite rings and finite abelian groups, and reminiscences of Francois Jaeger (a survey). Symposium à la Memoire de Francois Jaeger (Grenoble, 1998). Ann. Inst. Fourier (Grenoble) 49 (1999), no. 3, 763–782.

[2] Choie, Y. and Sole, P., Ternary codes and Jacobi forms. Discrete Math. 282 (2004), no. 1-3, 81–87.

[3] Conway, J.H, and Sloane, N. J. A. , Sphere Packings, Lattices and Groups (Second Edition), Springer-Verlag, NY, 1993.

[4] Duke, W., On codes and Siegel modular forms, Internat. Math. Res. Notices, 1993, no.5, 125–136.

[5] Igusa, J., Theta functions, Die Grundlehren der mathematischen Wissenschaften, Band 194, Springer-Verlag, NewYork-Heidelberg, 1972.

[6] Igusa, J., On the ring of modular forms of degree two over $\mathbf{Z}$, Amer. J. Math. 101 (1979), no. 1, 149–183.

[7] Lang, S., Introduction to Modular forms, A series of Comprehensive Studies in Mathematics, Springer-Verlag (1976).

[8] S. Nagaoka, On the ring of Hilbert modular forms over $\mathbf{Z}$, J.Math.Soc. Japan, Vol. 35, No. 4 (1983).

[9] Oura, M., Observation on the weight enumerators from classical invariant theory, preprint.

[10] Ozeki, M., On basis problem for Siegel moduar forms of degree 2, Acta Arith., **31** (1976), no.1, 17–30.

[11] Pless, V., Sloane, N. J. A., On the classification and enumeration of self-dual codes, J. Combinatorial Theory Ser.A **18** (1975), 313–335.

[12] Vardi, I., Coding theory (Multiple weight enumerators of codes), preprint 1998.

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