Centralizer algebras of the primitive unitary reflection group of order 96

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Abstract

Among the unitary reflection groups, the one on the title is singled out by its importance in, for example, coding theory and number theory. In this paper we start with describing the irreducible representations of this group and then examine the semi-simple structure of the centralizer algebra in the tensor representation.

1 Introduction

The group, which we denote by H_1, on the title of this paper consists of 96 matrices of size 2 by 2. It is the unitary group generated by reflections (u.g.g.r.), numbered as No.8 in Shephard-Todd [14]. This group, as well as No.9 in the same list, has long been recognized. The purpose of the present paper is to give a contribution to H_1 by decomposing the centralizer algebra of H_1 in the tensor representation into irreducible components.

We shall give an outline of the first statement in Abstract. The group H_1 naturally acts on the polynomial ring $\mathbb{C}[x,y]$ of 2 variables over the complex number field $\mathbb{C}$, i.e.

$$Af(x,y) = f(ax+by, cx+dy), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_1$$

for $f \in \mathbb{C}[x,y]$. We consider the invariant ring

$$\mathbb{C}[x,y]^{H_1} = \{f \in \mathbb{C}[x,y] : \quad Af = f, \quad \forall A \in H_1\}$$

of H_1. This ring has a rather simple structure. It is generated by two algebraically independent homogeneous polynomials of degrees 8 and 12, and conversely this nature characterizes the u.g.g.r. Broué-Enguehard [6] found a map

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connecting this invariant ring with number theory. Take a homogeneous polynomial $f(x, y)$ of degree n from the invariant ring. Introducing theta constants

$$\theta_{ab}(\tau) = \sum_{m \in \mathbb{Z}} \exp 2\pi i \left[\frac{1}{2}\tau \left(m + \frac{a}{2}\right)^2 + \left(m + \frac{a}{2}\right) \frac{b}{2} \right],$$

we get a modular form $f(\theta_{00}(2\tau), \theta_{10}(2\tau))$ of weight $n/2$ for $SL(2, \mathbb{Z})$. Moreover this map is an isomorphism from the invariant ring of H_1 onto the ring of modular forms for $SL(2, \mathbb{Z})$.

Next we proceed to coding theory. Let $\mathbb{F}_2 = \{0, 1\}$ be the field of two elements and $\mathbb{F}_2^n$ the vector space of dimension n over $\mathbb{F}_2$ equipped with the usual inner product $(u, v) = u_1 v_1 + \cdots + u_n v_n$. The weight of a vector u is the number of non-zero coordinates of u. A code of length n is by definition a linear subspace of $\mathbb{F}_2^n$. We impose two conditions on codes. The first one is the self-duality which says that a code C coincides with its dual code $C^\perp$, that is, $C = C^\perp$ in which

$$C^\perp = \{u \in \mathbb{F}_2^n : \ (u, v) = 0, \quad \forall v \in C\}.$$

The second one is the doubly-evenness which means

$$wt(u) \equiv 0 \pmod 4, \quad \forall u \in C.$$

These two notions give rise to the relation with invariant theory via the weight enumerator

$$W_C(x, y) = \sum_{v \in C} x^{n - wt(v)} y^{wt(v)}$$

of a code C. In fact, if C is self-dual, we have

$$W_C((x - y)/\sqrt{2}, (x + y)/\sqrt{2}) = W_C(x, y)$$

and if C is doubly even, we have

$$W_C(x, iy) = W_C(x, y).$$

We mention that a self-dual and doubly even code of length n exists if and only if n is a multiple of 8.

Now we can state the connections among all what we have mentioned. Take a positive integer $n \equiv 0 \pmod 8$. The weight enumerator of a self-dual doubly even code of length n is an invariant of H_1 and

$$W_C(\theta_{00}(2\tau), \theta_{10}(2\tau))$$

is a modular form of weight $n/2$ for $SL(2, \mathbb{Z})$. Gleason [9] showed that the invariants of degree n can be spanned by the weight enumerators of self-dual doubly even codes of length n. Finally any modular form of weight $n/2$ can be obtained from the weight enumerator of self-dual doubly even codes of length

k. The whole theory with more general results could be found in [11], [12] from which our notation H_1 comes.

Besides the importance of H_1, the motivation of this paper could be found in Brauer [5], Weyl [16]. One of the main ingredients there is the *commutator algebra* where invariant theory comes into play. We follow Weyl. Given any group of linear transformations in an n-dimensional space. Take covariant vectors $y^{(1)},\ldots,y^{(f)}$ and contravariant vectors $\xi^{(1)},\ldots,\xi^{(f)}$. A linear transformation acts on covariant vectors *cogrediently* and on contravariant vectors *contragradiently*. Then the matrices $\|b(i_1\cdots i_f;\ k_1\cdots k_f)\|$ in the tensor space obtained from the invariants

$$\sum_{i;k} b(i_1\cdots i_f;\ k_1\cdots k_f)\xi^{(1)}_{i_1}\cdots\xi^{(f)}_{i_f}y^{(1)}_{k_1}\cdots y^{(f)}_{k_f}$$

form the commutator algebra of H_1 in the tensor representation. The problem here is to decompose this algebra into simple parts. It is quite natural to apply this philosophy to our group H_1 as we will in this paper (*cf.* [1]).

2 Irreducible representations of H_1

In this section we determine the irreducible representations of H_1 which yields the character table. At the end of this section we discuss invariant theory of H_1 under the irreducible representations.

The unitary reflection group H_1 is a finite group in U_2 generated by the following matrices T and D:

$$T=\frac{1+i}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\epsilon & \epsilon\\ \epsilon & \epsilon^5\end{pmatrix},\quad D=\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}.$$

Here $\epsilon=\exp(2\pi i/8)$. It is known that the group size of H_1 is 96 and it has 16 conjugacy classes $\mathfrak{C}_1,\ldots,\mathfrak{C}_{16}$. Each conjugacy class has the following representative:

$$\mathfrak{C}_1\ni 1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \mathfrak{C}_2\ni T=\tfrac{1}{\sqrt{2}}\begin{pmatrix}\epsilon & \epsilon\\ \epsilon & \epsilon^5\end{pmatrix},\ \mathfrak{C}_3\ni T^2=\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix},$$
$$\mathfrak{C}_4\ni T^3=\tfrac{1}{\sqrt{2}}\begin{pmatrix}\epsilon^3 & \epsilon^3\\ \epsilon^3 & \epsilon^7\end{pmatrix},\ \mathfrak{C}_5\ni T^4=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},\ \mathfrak{C}_6\ni T^6=\begin{pmatrix}-i & 0\\ 0 & -i\end{pmatrix},$$
$$\mathfrak{C}_7\ni D=\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix},\ \mathfrak{C}_8\ni DT=\tfrac{1}{\sqrt{2}}\begin{pmatrix}\epsilon & \epsilon\\ \epsilon^3 & \epsilon^7\end{pmatrix},\ \mathfrak{C}_9\ni DT^2=\begin{pmatrix}i & 0\\ 0 & -1\end{pmatrix},$$
$$\mathfrak{C}_{10}\ni DT^3=\tfrac{1}{\sqrt{2}}\begin{pmatrix}\epsilon^3 & \epsilon^3\\ \epsilon^5 & \epsilon\end{pmatrix},\ \mathfrak{C}_{11}\ni DT^4=\begin{pmatrix}-1 & 0\\ 0 & -i\end{pmatrix},$$
$$\mathfrak{C}_{12}\ni DT^5=\tfrac{1}{\sqrt{2}}\begin{pmatrix}\epsilon^5 & \epsilon^5\\ \epsilon^7 & \epsilon^3\end{pmatrix},\ \mathfrak{C}_{13}\ni DT^6=\begin{pmatrix}-i & 0\\ 0 & 1\end{pmatrix},$$
$$\mathfrak{C}_{14}\ni DT^7=\tfrac{1}{\sqrt{2}}\begin{pmatrix}\epsilon^7 & \epsilon^7\\ \epsilon & \epsilon^5\end{pmatrix},\ \mathfrak{C}_{15}\ni D^2=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \mathfrak{C}_{16}\ni D^2T^2=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}.$$

Since the number of conjugacy classes and that of the non-isomorphic irreducible representations coincide, there exist 16 classes of the irreducible representations of H_1. In the following, we construct all of them one by one.

First we note that any group has the trivial representation which maps each element of the group to 1. We denote that of H_1 by (ρ_1, V_1). The determinant which maps T and D to $-i$ and i respectively also gives a one-dimensional irreducible representation. We call it (ρ_3, V_3). The tensor product $\rho_3^{\otimes 2}$ also gives a one-dimensional irreducible representation, which maps both T and D to -1. We name it (ρ_2, V_2). Also $\rho_2 \otimes \rho_3$ defines a one-dimensional representation. We name it (ρ_4, V_4).

Next we consider two-dimensional representations. The natural representation (ρ_{10}, V_{10}) which maps T and D to the defining matrices above is irreducible, since neither of one-dimensional D-invariant subspaces are T-invariant. Taking tensor products with the one-dimensional representations above and the natural representation, we have further 3 two-dimensional irreducible representations, $\rho_7 = \rho_3 \otimes \rho_{10}$, $\rho_8 = \rho_2 \otimes \rho_{10}$ and $\rho_9 = \rho_4 \otimes \rho_{10}$. There are 2 more two-dimensional irreducible representations which we will deal with later.

As a subrepresentation of $\rho_{10} \otimes \rho_{10}$, we have a three-dimensional irreducible representation. Let $\langle \boldsymbol{e}_1, \boldsymbol{e}_2 \rangle$ be a basis of V_{10} which gives the natural representation. Then $\langle \boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \boldsymbol{e}_1 \otimes \boldsymbol{e}_2, \boldsymbol{e}_2 \otimes \boldsymbol{e}_1, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \rangle$ gives a basis for the tensor representation $\rho_{10}^{\otimes 2}$. With respect to this basis, the representation matrices of T and D are

$$\rho_{10}^{\otimes 2}(T) = \frac{i}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \rho_{10}^{\otimes 2}(D) = \operatorname{diag}(1, i, i, -1).$$

If we put $\boldsymbol{e}'_1 = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1$, $\boldsymbol{e}'_2 = \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1$ and $\boldsymbol{e}'_3 = \boldsymbol{e}_2 \otimes \boldsymbol{e}_2$, then $\langle \boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3 \rangle$ is obviously a D-invariant subspace. It is easy to check that it is also T-invariant. Hence it gives a three-dimensional representation. We name it (ρ_{13}, V_{13}). The representation matrices with respect to this basis are

$$\rho_{13}(T) = \frac{i}{2} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix} \text{ and } \rho_{13}(D) = \operatorname{diag}(1, i, -1).$$

Since each one-dimensional D-invariant subspace of V_{13} is not T-invariant, the representation (ρ_{13}, V_{13}) is irreducible. Similarly to the previous case, we have further 3 three-dimensional irreducible representations, $\rho_{11} = \rho_3 \otimes \rho_{13}$, $\rho_{12} = \rho_4 \otimes \rho_{13}$ and $\rho_{14} = \rho_2 \otimes \rho_{13}$.

Next we look for a four-dimensional irreducible representation in $(\rho_{10} \otimes \rho_{13}, V_{10} \otimes V_{13})$. Let $\langle \boldsymbol{e}_i \otimes \boldsymbol{e}'_j \mid i = 1, 2,\ j = 1, 2, 3 \rangle$ be a basis of $V_{10} \otimes V_{13}$ (lexicographical order). Then we have the following representation matrices of

T and D:

$$\rho_{10} \otimes \rho_{13}(T) = \frac{-1+i}{4} \begin{pmatrix} 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 1 & 0 & -1 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 1 & 2 & 1 & -1 & -2 & -1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & -2 & 1 & -1 & 2 & -1 \end{pmatrix},$$

$$\rho_{10} \otimes \rho_{13}(D) = \mathrm{diag}(1, i, -1, i, -1, -i).$$

If we put $\boldsymbol{e}_1'' = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1'$, $\boldsymbol{e}_2'' = \boldsymbol{e}_1 \otimes \boldsymbol{e}_2' + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1'$, $\boldsymbol{e}_3'' = \boldsymbol{e}_1 \otimes \boldsymbol{e}_3' + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2'$, and $\boldsymbol{e}_4'' = \boldsymbol{e}_2 \otimes \boldsymbol{e}_3'$, according to the eigen values of $\rho_{10} \otimes \rho_{13}(D)$, then $\langle \boldsymbol{e}_k'' \mid k = 1, 2, 3, 4\rangle$ is obviously a D-invariant subspace. It is also easy to check that it is T-invariant. Hence it gives a four-dimensional representation. We name it $(\rho_{15},\, V_{15})$. The representation matrices with respect to this basis are

$$\rho_{15}(T) = \frac{-1+i}{4} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix} \text{ and } \rho_{13}(D) = \mathrm{diag}(1, i, -1, -i).$$

As we saw in the previous case, none of one-dimensional D-invariant subspaces of V_{15} is T-invariant. Now consider two-dimensional D-invariant subspaces. Since all eigen spaces of $\rho_{15}(D)$ are one-dimensional, we find that a two-dimensional D-invariant subspace is of the form $\langle \boldsymbol{e}_i'', \boldsymbol{e}_j''\rangle (i \neq j)$. Let W be $\langle \boldsymbol{e}_1'', \boldsymbol{e}_2''\rangle$ and take a non-zero vector $\boldsymbol{v} = a\boldsymbol{e}_1'' + b\boldsymbol{e}_2''$ from W. Then we have

$$\rho_{15}(T)\boldsymbol{v} = \frac{-1+i}{4} \left[(a+3b)\boldsymbol{e}_1'' + (a+b)\boldsymbol{e}_2'' + (a-b)\boldsymbol{e}_3'' + (a-3b)\boldsymbol{e}_4''\right].$$

In order that $\rho_{15}(T)\boldsymbol{v} \in W$, it must hold that $a = b = 0$. This contradicts the assumption that $\boldsymbol{v}$ is non-zero vector. Hence we find that W is not T-invariant. Similar arguments hold for all two-dimensional D-invariant subspaces $\{\langle \boldsymbol{e}_i'', \boldsymbol{e}_j''\rangle\}_{1 \leq i < j \leq 4}$. This implies there is no two-dimensional subrepresentation in V_{15}. Hence we find that $(\rho_{15},\, V_{15})$ is irreducible. Similarly to the previous case, we have further four-dimensional irreducible representations, $\rho_2 \otimes \rho_{15}$, $\rho_3 \otimes \rho_{15}$, $\rho_4 \otimes \rho_{15}$. The first one, however, coincides with ρ_{15} and the second one and the third one are equivalent. Hence we have 2 four-dimensional irreducible representations, ρ_{15} and $\rho_{16} = \rho_3 \otimes \rho_{15}$.

Finally we look for the remaining irreducible representations in $(\rho_{10} \otimes \rho_{15}, V_{10} \otimes V_{15})$. Let $\langle \boldsymbol{e}_i \otimes \boldsymbol{e}_j'' \mid i = 1, 2, j = 1, 2, 3, 4\rangle$ be a basis of $V_{10} \otimes V_{15}$ (lexicographical order). Then we have the following representation matrices of T and D:

$$\rho_{10} \otimes \rho_{15}(T) = \frac{-1}{4} \begin{pmatrix} 1 & 3 & 3 & 1 & 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 & 1 & -3 & 3 & -1 \\ 1 & 3 & 3 & 1 & -1 & -3 & -3 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -3 & 3 & -1 & -1 & 3 & -3 & 1 \end{pmatrix},$$

$$\rho_{10} \otimes \rho_{15}(D) = \mathrm{diag}(1, i, -1, -i, i, -1, -i, 1).$$

If we put $\boldsymbol{e}_1''' = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1'' + \boldsymbol{e}_2 \otimes \boldsymbol{e}_4''$, and $\boldsymbol{e}_2''' = \boldsymbol{e}_1 \otimes \boldsymbol{e}_3'' + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2''$, then $\langle \boldsymbol{e}_1''', \boldsymbol{e}_2''' \rangle$ is T- and D-invariant subspace. We name it (ρ_5, V_5). The representation matrices with respect to this basis are

$$\rho_5(T) = \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \text{ and } \rho_5(D) = \mathrm{diag}(1, -1).$$

Similarly to the previous ones, we can check that this representation is irreducible. Further $\rho_6 = \rho_3 \otimes \rho_5$ also defines a two-dimensional irreducible representation.

So far, we have got 16 irreducible representations. Since H_1 has 16 conjugacy classes, $\{(\rho_i, V_i)\}_{i=1}^{16}$ are a complete representatives of all irreducible representations of H_1. Accordingly, the character table of H_1 is also derived.

H_1	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$	$\mathfrak{C}_6$	$\mathfrak{C}_7$	$\mathfrak{C}_8$	$\mathfrak{C}_9$	$\mathfrak{C}_{10}$	$\mathfrak{C}_{11}$	$\mathfrak{C}_{12}$	$\mathfrak{C}_{13}$	$\mathfrak{C}_{14}$	$\mathfrak{C}_{15}$	$\mathfrak{C}_{16}$
	1	T	T^2	T^3	T^4	T^6	D	DT	DT^2	DT^3	DT^4	DT^5	DT^6	DT^7	D^2	D^2T^2
order	1	8	4	8	2	4	4	6	4	12	4	3	4	12	2	4
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	1	1
χ_3	1	$-i$	-1	i	1	-1	i	1	$-i$	-1	i	1	$-i$	-1	-1	1
χ_4	1	i	-1	$-i$	1	-1	$-i$	1	i	-1	$-i$	1	i	-1	-1	1
χ_5	2	0	2	0	2	2	0	-1	0	-1	0	-1	0	-1	2	2
χ_6	2	0	-2	0	2	-2	0	-1	0	1	0	-1	0	1	-2	2
χ_7	2	0	$-2i$	0	-2	$2i$	$-1+i$	1	$1+i$	$-i$	$1-i$	-1	$-1-i$	i	0	0
χ_8	2	0	$2i$	0	-2	$-2i$	$-1-i$	1	$1-i$	i	$1+i$	-1	$-1+i$	$-i$	0	0
χ_9	2	0	$-2i$	0	-2	$2i$	$1-i$	1	$-1-i$	$-i$	$-1+i$	-1	$1+i$	i	0	0
χ_{10}	2	0	$2i$	0	-2	$-2i$	$1+i$	1	$-1+i$	i	$-1-i$	-1	$1-i$	$-i$	0	0
χ_{11}	3	1	3	1	3	3	-1	0	-1	0	-1	0	-1	0	-1	-1
χ_{12}	3	-1	3	-1	3	3	1	0	1	0	1	0	1	0	-1	-1
χ_{13}	3	i	-3	$-i$	3	-3	i	0	$-i$	0	i	0	$-i$	0	1	-1
χ_{14}	3	$-i$	-3	i	3	-3	$-i$	0	i	0	$-i$	0	i	0	1	-1
χ_{15}	4	0	$-4i$	0	-4	$4i$	0	-1	0	i	0	1	0	$-i$	0	0
χ_{16}	4	0	$4i$	0	-4	$-4i$	0	-1	0	$-i$	0	1	0	i	0	0

We conclude this section with adding a few words on invariant theory of H_1 under irreducible representations (*cf.* [8]). Let ρ be one of the d-dimensional irreducible representation of H_1. Then $\rho(H_1)$ acts naturally on the polynomial ring of d variables. We denote the invariant ring under this action by $\mathbb{C}[\rho]^{H_1}$. The orders of $\rho_i(H_1)$ are

$$\underbrace{1,2,4,4}_{\text{dim }1},\underbrace{6,12,96,96,96,96}_{\text{dim }2},\underbrace{24,24,48,48}_{\text{dim }3},\underbrace{96,96}_{\text{dim }4}.$$

The dimension 1 case aside, the invariant rings

$$\mathbb{C}[\rho_5]^{H_1}, \mathbb{C}[\rho_i]^{H_1}\ (i=7,8,9,10), \mathbb{C}[\rho_{12}]^{H_1}$$

are weighted polynomial rings. In the sense of [14], all $\rho_i(H_1)$ $(i=7,8,9,10)$ are equivalent each other, and $\rho_{15}(H_1)$ to $\rho_{16}(H_1)$. We already know the ring $\mathbb{C}[\rho_7]^{H_1}$. The ring $\mathbb{C}[\rho_5]^{H_1}$ can be generated by the polynomials of degrees 2 and 3, and the ring $\mathbb{C}[\rho_{15}]^{H_1}$ by those of degrees $2,3$ and 4. If we look at the degrees, we can find that $\rho_5(H_1)$ is equivalent to $G(3,3,2)$ and $\rho_{12}(H_1)$ to $G(2,2,3)$. The other cases up to dimension 3 are modules of rank 2 over the polynomial rings. The ρ_{15} case has a somewhat complicated structure. The ring $\mathbb{C}[\rho_{15}]^{H_1}$ is a module of rank 32 over the polynomial ring. We note that calculations here were done with Magma [4].

3 Decomposition of tensor representations

In the previous section, we have found complete representatives of all irreducible representations. In this section, we see how tensor powers of ρ_{10} are decomposed into irreducible ones.

We begin with the general theory (see for example Curtis-Reiner[7]). Let $\chi_1,\ldots,\chi_s$ be the set of all irreducible characters of a finite group G. For any (not necessarily irreducible) representation (ρ,V) of G, let χ be its character. Then χ can be uniquely expressed a sum of irreducible characters:

$$\chi = m_1\chi_1 + \cdots + m_s\chi_s.$$

Now suppose that χ has its character values $(k_1,\ldots,k_s)$ on the conjugacy classes $(\mathfrak{C}_1,\ldots,\mathfrak{C}_s)$. Then we get

$$\begin{aligned}(k_1,\ldots,k_s) &= (\chi(\mathfrak{C}_1),\ldots,\chi(\mathfrak{C}_s))\\ &= m_1(\chi_1(\mathfrak{C}_1),\ldots,\chi_s(\mathfrak{C}_s)) + \cdots + m_s(\chi_s(\mathfrak{C}_1),\ldots,\chi_s(\mathfrak{C}_s))\\ &= (m_1,\ldots,m_s)\begin{pmatrix}\chi_1(\mathfrak{C}_1) & \cdots & \chi_1(\mathfrak{C}_s)\\ \vdots & \ddots & \vdots\\ \chi_s(\mathfrak{C}_1) & \cdots & \chi_s(\mathfrak{C}_s)\end{pmatrix}.\end{aligned}$$

If we let $\boldsymbol{X}$ denote the matrix of the character table, then the above relation is simply written as

$$(k_1,\ldots,k_s) = (m_1,\ldots,m_s)\boldsymbol{X}. \tag{1}$$

By the linear independence of the irreducible characters, $\boldsymbol{X}$ is non-singular. Hence we have

$$(m_1, \ldots, m_s) = (k_1, \ldots, k_s)\boldsymbol{X}^{-1}. \tag{2}$$

In order to examine the structure of the centralizer algebra of the tensor representation, it is useful to investigate how the tensor product of the natural and an irreducible representation is decomposed into the irreducible ones. In the following, we go back to our case and decompose $\rho_{10} \otimes \rho_i$ $(i = 1, 2, \ldots, 16)$ one by one.

By the argument and/or the character table in the previous section, we already have the following:

$$\begin{aligned}
\chi_{10} \cdot \chi_1 &= \chi_{10}, \\
\chi_{10} \cdot \chi_2 &= \chi_8, \\
\chi_{10} \cdot \chi_3 &= \chi_7, \\
\chi_{10} \cdot \chi_4 &= \chi_9.
\end{aligned}$$

Further, we can directly read the following from the character table:

$$\begin{aligned}
\chi_{10} \cdot \chi_5 &= \chi_{16}, \\
\chi_{10} \cdot \chi_6 &= \chi_{15}.
\end{aligned}$$

Next, consider $\chi_{10} \cdot \chi_7(\mathfrak{C}_1, \ldots, \mathfrak{C}_{16})$. Again from the character table, we have

$$\begin{aligned}
&\chi_{10} \cdot \chi_7(\mathfrak{C}_1, \ldots, \mathfrak{C}_{16}) \\
&= (4, 0, 4, 0, 4, 4, -2, 1, -2, 1, -2, 1, -2, 1, 0, 0).
\end{aligned}$$

Using the identity (2), we have

$$\begin{aligned}
&(0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0) \\
&= \chi_{10} \cdot \chi_7(\mathfrak{C}_1, \ldots, \mathfrak{C}_{16})\boldsymbol{X}^{-1}.
\end{aligned}$$

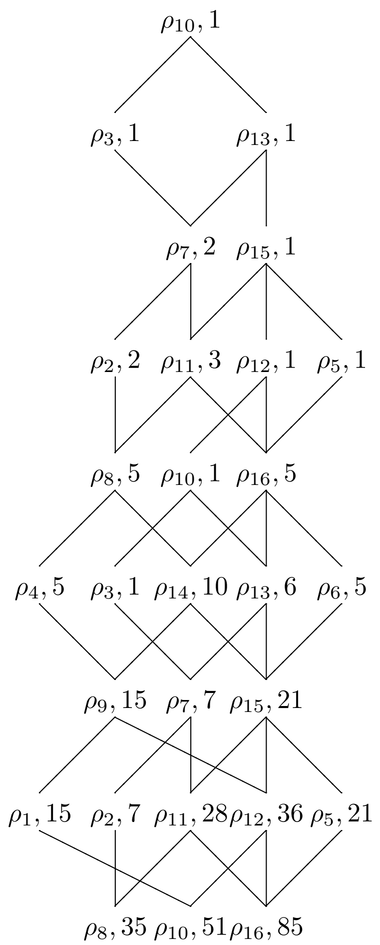
This means

$$\chi_{10} \cdot \chi_7 = \chi_2 + \chi_{11}.$$

In a similar way we have

$$\begin{aligned}
\chi \cdot \chi_8 &= \chi_4 + \chi_{14}, \\
\chi \cdot \chi_9 &= \chi_1 + \chi_{12}, \\
\chi \cdot \chi_{10} &= \chi_3 + \chi_{13}, \\
\chi \cdot \chi_{11} &= \chi_8 + \chi_{16}, \\
\chi \cdot \chi_{12} &= \chi_{10} + \chi_{16}, \\
\chi \cdot \chi_{13} &= \chi_7 + \chi_{15}, \\
\chi \cdot \chi_{14} &= \chi_9 + \chi_{15}, \\
\chi \cdot \chi_{15} &= \chi_5 + \chi_{11} + \chi_{12}, \\
\chi \cdot \chi_{16} &= \chi_6 + \chi_{13} + \chi_{14}.
\end{aligned}$$

By the above calculation, we obtain the Hasse diagram of the decomposition of $\rho_{10}^{\otimes k}$ into irreducible ones.



$1^2 = 1$

$1^2 + 1^2 = 2$

$2^2 + 1^2 = 5$

$2^2 + 3^2 + 1^2 + 1^2 = 15$

$5^2 + 1^2 + 5^2 = 51$

$5^2 + 1^2 + 10^2 + 6^2 + 5^2 = 187$

$15^2 + 7^2 + 21^2 = 715$

$15^2 + 7^2 + 28^2 + 36^2 + 21^2 = 2795$

$35^2 + 51^2 + 85^2 = 11051$

From this diagram we can read off the multiplicity of each irreducible representation (ρ_i, V_i) in $V_{10}^{\otimes k}$ by counting the number of paths from the top vertex indexed by ρ_{10} in the 1-st row to the corresponding vertex in the k-th row. We put the multiplicity on the right side of each irreducible representation. Further, we calculated the square sums of the multiplicities on the each row.

Let $\mathcal{A}_k = \mathrm{End}_{H_1}(V_{10}^{\otimes k})$ be the centralizer algebra of H_1 in $V_{10}^{\otimes k}$, where H_1 acts on V_{10} diagonally. By the Schur-Weyl reciprocity [13, 16], this diagram is the Bratteli diagram of the algebra sequence

$$\mathbb{C} = \mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \cdots .$$

(For the Bratteli diagram, see for example Goodman-de la Harpe-Jones [10], §2.3.) Accordingly, the square sum of the multiplicities on the k-th row is the dimension of $\mathrm{End}_{H_1}(V_{10}^{\otimes k})$. We will examine it in detail in the next section.

4 Centralizer algebra

In the previous section, we have seen that the dimensions of $\mathcal{A}_k = \mathrm{End}_{H_1}(V_{10}^{\otimes k})$ $(k = 1, 2, \ldots)$ are 1, 2, 5, 15, 51, 187, 715,.... According to "The On-Line Encyclopedia of Integer Sequences"[15], these terms coincide with the fist few terms of the expression $(3 \cdot 2^{k-2} + 2^{2k-3} + 1)/3$. This is indeed the case. In order to prove this, we calculate the size of each simple component of $\mathcal{A}_k$.

Let $d_j^{(i)}$ be the multiplicity of ρ_i in the tensor representation ρ_{10}^j, which coincides with the size of the corresponding simple component of $\mathcal{A}_j$. By the Bratteli diagram of $\mathcal{A}_j$ given in the previous section, we have the recursive formulae as follows. First note that the irreducible representations ρ_8, ρ_{10}, ρ_{16} of H_1 again appear in the bottom of the diagram, as well as the 5-th row of the diagram. This implies that the diagram periodically grows up as k increases. The iteration is as follows:

$$[\rho_8, \rho_{10}, \rho_{16}] \to [\rho_4, \rho_3, \rho_{14}, \rho_{13}, \rho_6] \to [\rho_9, \rho_7, \rho_{15}] \to [\rho_1, \rho_2, \rho_{11}, \rho_{12}, \rho_5] \to \cdots.$$

Hence based on the Bratteli diagram of the 9-th row from the 5-th row, we can obtain the following recursive formulae:

$$\begin{cases} d_{4\ell+2}^{(4)} &= d_{4\ell+1}^{(8)}, \\ d_{4\ell+2}^{(3)} &= d_{4\ell+1}^{(10)}, \\ d_{4\ell+2}^{(14)} &= d_{4\ell+1}^{(8)} + d_{4\ell+1}^{(16)}, \\ d_{4\ell+2}^{(13)} &= d_{4\ell+1}^{(10)} + d_{4\ell+1}^{(16)}, \\ d_{4\ell+2}^{(6)} &= d_{4\ell+1}^{(16)}, \end{cases}$$

$$\begin{cases} d_{4\ell+3}^{(9)} &= d_{4\ell+2}^{(4)} + d_{4\ell+2}^{(14)}, \\ d_{4\ell+3}^{(7)} &= d_{4\ell+2}^{(3)} + d_{4\ell+2}^{(13)}, \\ d_{4\ell+3}^{(15)} &= d_{4\ell+2}^{(14)} + d_{4\ell+2}^{(13)} + d_{4\ell+2}^{(6)}, \end{cases}$$

$$\begin{cases} d_{4(\ell+1)}^{(1)} &= d_{4\ell+3}^{(9)}, \\ d_{4(\ell+1)}^{(2)} &= d_{4\ell+3}^{(7)}, \\ d_{4(\ell+1)}^{(11)} &= d_{4\ell+3}^{(7)} + d_{4\ell+3}^{(15)}, \\ d_{4(\ell+1)}^{(12)} &= d_{4\ell+3}^{(9)} + d_{4\ell+3}^{(15)}, \\ d_{4(\ell+1)}^{(5)} &= d_{4\ell+3}^{(15)}, \end{cases}$$

$$\begin{cases} d_{4(\ell+1)+1}^{(8)} &= d_{4(\ell+1)}^{(2)} + d_{4(\ell+1)}^{(11)}, \\ d_{4(\ell+1)+1}^{(10)} &= d_{4(\ell+1)}^{(1)} + d_{4(\ell+1)}^{(12)}, \\ d_{4(\ell+1)+1}^{(16)} &= d_{4(\ell+1)}^{(11)} + d_{4(\ell+1)}^{(12)} + d_{4(\ell+1)}^{(5)}. \end{cases}$$

Note also that if we allow the possibility $d_j^{(i)} = 0$, the recursions above are still valid even for the 1-st to 4-th row. Hence we have the following:

$$\begin{aligned}
d^{(8)}_{4\ell+1} &= d^{(2)}_{4\ell} + d^{(11)}_{4\ell} \\
&= d^{(7)}_{4\ell-1} + (d^{(7)}_{4\ell-1} + d^{(15)}_{4\ell-1}) \\
&= 2d^{(7)}_{4\ell-1} + d^{(15)}_{4\ell-1} \\
&= 2(d^{(3)}_{4\ell-2} + d^{(13)}_{4\ell-2}) + (d^{(14)}_{4\ell-2} + d^{(13)}_{4\ell-2} + d^{(6)}_{4\ell-2}) \\
&= 2d^{(3)}_{4\ell-2} + d^{(14)}_{4\ell-2} + 3d^{(13)}_{4\ell-2} + d^{(6)}_{4\ell-2} \\
&= 2d^{(10)}_{4\ell-3} + (d^{(8)}_{4\ell-3} + d^{(16)}_{4\ell-3}) + 3(d^{(10)}_{4\ell-3} + d^{(16)}_{4\ell-3}) + d^{(16)}_{4\ell-3} \\
&= d^{(8)}_{4(\ell-1)+1} + 5d^{(10)}_{4(\ell-1)+1} + 5d^{(16)}_{4(\ell-1)+1} \ (\ell > 0). \qquad (3)
\end{aligned}$$

Similarly we have

$$d^{(10)}_{4\ell+1} = 5d^{(8)}_{4(\ell-1)+1} + d^{(10)}_{4(\ell-1)+1} + 5d^{(16)}_{4(\ell-1)+1} \qquad (4)$$

and

$$d^{(16)}_{4\ell+1} = 5d^{(8)}_{4(\ell-1)+1} + 5d^{(10)}_{4(\ell-1)+1} + 11d^{(16)}_{4(\ell-1)+1} \qquad (5)$$

From the recursion (3), (4) and (5), and the initial condition $(d^{(8)}_1, d^{(10)}_1, d^{(16)}_1) = (0, 1, 0)$, we obtain

$$\begin{aligned}
d^{(8)}_{4\ell+1} &= -\frac{(-4)^\ell}{2} + \frac{1}{3} + \frac{16^\ell}{6}, \\
d^{(10)}_{4\ell+1} &= \frac{(-4)^\ell}{2} + \frac{1}{3} + \frac{16^\ell}{6}, \\
d^{(16)}_{4\ell+1} &= -\frac{1}{3} + \frac{16^\ell}{3}.
\end{aligned}$$

By the initial recursion formulae, we immediately obtain

$$\begin{aligned}
d^{(4)}_{4\ell+2} &= -\frac{(-4)^\ell}{2} + \frac{1}{3} + \frac{16^\ell}{6}, \\
d^{(3)}_{4\ell+2} &= \frac{(-4)^\ell}{2} + \frac{1}{3} + \frac{16^\ell}{6}, \\
d^{(14)}_{4\ell+2} &= -\frac{(-4)^\ell}{2} + \frac{16^\ell}{2}, \\
d^{(13)}_{4\ell+2} &= \frac{(-4)^\ell}{2} + \frac{16^\ell}{2}, \\
d^{(6)}_{4\ell+2} &= -\frac{1}{3} + \frac{16^\ell}{3},
\end{aligned}$$

$$d^{(9)}_{4\ell+3} = -(-4)^{\ell} + \frac{1}{3} + \frac{2 \cdot 16^{\ell}}{3},$$
$$d^{(7)}_{4\ell+3} = (-4)^{\ell} + \frac{1}{3} + \frac{2 \cdot 16^{\ell}}{3},$$
$$d^{(15)}_{4\ell+3} = -\frac{1}{3} + \frac{4 \cdot 16^{\ell}}{3}$$

and

$$d^{(1)}_{4(\ell+1)} = -(-4)^{\ell} + \frac{1}{3} + \frac{2 \cdot 16^{\ell}}{3},$$
$$d^{(2)}_{4(\ell+1)} = (-4)^{\ell} + \frac{1}{3} + \frac{2 \cdot 16^{\ell}}{3},$$
$$d^{(11)}_{4(\ell+1)} = (-4)^{\ell} + 2 \cdot 16^{\ell},$$
$$d^{(12)}_{4(\ell+1)} = -(-4)^{\ell} + 2 \cdot 16^{\ell},$$
$$d^{(5)}_{4(\ell+1)} = -\frac{1}{3} + \frac{4 \cdot 16^{\ell}}{3}.$$

Thus we have obtained the size of each simple component of $\mathcal{A}_k$. If we apply simple considerations to the order of simple components, the sizes are uniformly described as follows.

Theorm 4.1. *Let* $\mathcal{A}_k = End_{H_1}(V_{10}^{\otimes k})$ *be a centralizer algebra of* H_1 *in* $V_{10}^{\otimes k}$, *where* H_1 *acts on* V_{10} *diagonally. Then* $\mathcal{A}_k$ *has the following multi-matrix structure.*

$$\mathcal{A}_k \cong \begin{cases} M_{d_+(k)}(\mathbb{C}) \oplus M_{d_-(k)}(\mathbb{C}) \oplus M_{d_0(k)}(\mathbb{C}) \\ \quad \text{if } k = 2m-1, \\ M_{d_+(k)}(\mathbb{C}) \oplus M_{d_-(k)}(\mathbb{C}) \oplus M_{d_0(k)}(\mathbb{C}) \oplus M_{e_+(k)}(\mathbb{C}) \oplus M_{e_-(k)}(\mathbb{C}) \\ \quad \text{if } k = 2m, \end{cases}$$

where

$$d_{\pm}(k) = \pm 2^{m-2} + \frac{1}{3} + \frac{2 \cdot 4^{m-2}}{3},$$
$$d_0(k) = -\frac{1}{3} + \frac{4^{m-1}}{3}$$

and

$$e_{\pm} = \pm 2^{m-2} + 2 \cdot 4^{m-2}.$$

Calculating the square sum of the dimensions of the simple components of $\mathcal{A}_k$ in cases $k = 2m-1$ and $k = 2m$, we finally obtain the following corollary as we expected.

Corollary 4.2.

$$\dim \mathcal{A}_k = 2^{k-2} + \frac{2^{2k-3}}{3} + \frac{1}{3}.$$

Again by the web cite [15], this result suggests that the basis of the centralizer algebras of H_1 could be described in terms of the symmetric polynomials in 4 noncommuting variables [2] and/or the universal embedding of the symplectic dual polar space $DS_p(2k, 2)$ [3]. It would be interesting that these points become clear.

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