On Eisenstein polynomials and zeta polynomials II *

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Abstract

Eisenstein polynomials, which were defined by the second named author, are analogues of the concept of an Eisenstein series. The second named author conjectured that there exist some analogous properties between Eisenstein series and Eisenstein polynomials. In the previous paper, the first named author provided new analogous properties of Eisenstein polynomials and zeta polynomials for Type II case. In this paper, the analogous properties of Eisenstein polynomials and zeta polynomials also hold for the cases Type I, Type III, and Type IV. These properties are finite analogies of certain properties of Eisenstein series.

Key Words: Eisenstein polynomials, Zeta polynomials, Weight enumerators.

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1 Introduction

In the present paper, we discuss some analogies between Eisenstein series, Eisenstein polynomials, and zeta polynomials. This paper is a sequel to the previous paper [10]. To explain our results, we recall the paper [10].

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A linear code C of length n is a linear subspace of $\mathbb{F}_q^n$. Then, the dual $C^\perp$ of a linear code C is defined as follows: $C^\perp = \{\mathbf{y} \in \mathbb{F}_q^n \mid (\mathbf{x}, \mathbf{y}) = \mathbf{0} \text{ for all } \mathbf{x} \in C\}$. A linear code C is called self-dual if $C = C^\perp$. The weight $\mathrm{wt}(\mathbf{x})$ is the number of its nonzero components. The weight enumerator of a code C is

$$w_C(x,y) = \sum_{\mathbf{u} \in C} x^{n-wt(\mathbf{u})} y^{wt(\mathbf{u})} = x^n + \sum_{i=1}^{n} A_i x^{n-i} y^i,$$

where A_i is the number of codewords of weight i. In this paper, we consider the following self-dual codes, [1]:

Type I: Over $\mathbb{F}_2^n$ with all weights divisible by 2,
Type II: Over $\mathbb{F}_2^n$ with all weights divisible by 4,
Type III: Over $\mathbb{F}_3^n$ with all weights divisible by 3,
Type IV: Over $\mathbb{F}_4^n$ with all weights divisible by 2.

For Type I, …, IV, it is well known that the weight enumerator $w_C(x,y)$ is in the space $\mathbb{C}[f,g]$, [1], where

Type I : $f = x^2 + y^2$,$g = x^2y^2(x^2 - y^2)^2$,
Type II: $f = x^8 + 14x^4y^4 + y^8$,$g = x^4y^4(x^4 - y^4)^4$,
Type III: $f = x^4 + 8xy^3$,$g = y^3(x^3 - y^3)^3$,
Type IV: $f = x^2 + 3y^2$,$g = y^2(x^2 - y^2)^2$.

Let

Type I: $G_{\mathrm{I}} = \left\langle \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$,
Type II: $G_{\mathrm{II}} = \left\langle \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \right\rangle$,
Type III: $G_{\mathrm{III}} = \left\langle \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi\sqrt{-1}/3} \end{pmatrix} \right\rangle$,
Type IV: $G_{\mathrm{IV}} = \left\langle \frac{1}{2}\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$.

It is known that for $\mathrm{X} \in \{\mathrm{I}, \ldots, \mathrm{IV}\}$, a weight enumerator of Type X codes is an invariant polynomial of the group G_{X}, namely, for all $\sigma \in G_{\mathrm{X}}$,

$$w_C(\sigma(x,y)) = w_C(x,y),$$

where $\sigma(x,y) := \sigma^t\,(x,y)$. We denote by $\mathbb{C}[x,y]^{G_{\mathrm{X}}}$ the G_{X}-invariant subring of $\mathbb{C}[x,y]$.

Here is a typical example of $\mathbb{C}[x,y]^{G_{\mathrm{X}}}$. Oura defined an Eisenstein polynomial for Type II as follows:

$$\varphi_\ell^{G_{\mathrm{II}}}(x) = \frac{1}{|G_{\mathrm{II}}|}\sum_{\sigma\in G_{\mathrm{II}}}(\sigma x)^\ell.$$

[14, 16]. Here, we define an Eisenstein polynomial for Type X as follows:

$$\varphi_\ell^{G_{\mathrm{X}}}(x) = \frac{1}{|G_{\mathrm{X}}|}\sum_{\sigma\in G_{\mathrm{X}}}(\sigma x)^\ell.$$

It is straightforward to show that the Eisenstein polynomial for Type X is in $\mathbb{C}[x,y]^{G_{\mathrm{X}}}$.

We introduce an expression relating G_{II} and modular forms $M(\Gamma_1)$. For the detailed expression of modular forms, see [1, 5, 6, 8, 9]. For $\mathbb{C}[x,y]^{G_{\mathrm{X}}}$, we construct the elements of Γ_1 as follows:

$$\begin{aligned} Th : \mathbb{C}[x,y]^{G_{\mathrm{X}}} &\to M(\Gamma_1)\\ x \mapsto f_0(\tau) &= \sum_{b\in\mathbb{Z}, b\equiv 0 \pmod 2} \exp(\pi\sqrt{-1}\tau b^2/2),\\ y \mapsto f_1(\tau) &= \sum_{b\in\mathbb{Z}, b\equiv 1 \pmod 2} \exp(\pi\sqrt{-1}\tau b^2/2). \end{aligned}$$

The map Th is called the theta map.

Here is a typical example of $M(\Gamma_1)$. The Eisenstein series for Γ_1 is defined as follows:

$$\psi_k^{\Gamma_1}(z) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n,$$

where B_k is the k-th Bernoulli number and $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$. For the detailed expression of the Eisenstein series, see [1, 5, 6, 8, 9].

The elements of both $M(\Gamma_1)$ and $\mathbb{C}[x,y]^{G_{\mathrm{X}}}$ are “invariant functions” and the Eisenstein series and the Eisenstein polynomial are “average functions” of the groups. Therefore, these two objects are expected to have similar properties. Moreover, for $f \in \mathbb{C}[x,y]^{G_{\mathrm{X}}}$, it is expected that f and $Th(f)$ have similar properties. Table 1 shows a summary of the concepts that we have introduced so far.

For $\varphi_\ell^{G_{\mathrm{X}}}(x,y) \not\equiv 0$, we denote by $\widetilde{\varphi_\ell^{G_{\mathrm{X}}}}(x,y)$ the polynomial $\varphi_\ell^{G_{\mathrm{X}}}(x,y)$ divided by its x^ℓ coefficient. We give some examples in Table 2.

Table 1: Summary of our objects

Γ_1	G_{II}
$M(\Gamma_1)$	$\mathbb{C}[x,y]^{G_{\mathrm{II}}}$
Eisenstein series	Eisenstein polynomials
f	$Th(f)$

Table 2: Examples of Eisenstein polynomials

ℓ	$\widetilde{\varphi_\ell^{G_{\mathrm{II}}}}(x,y)$
8	$x^8+14x^4y^4+y^8$
12	$x^{12}-33x^8y^4-33x^4y^8+y^{12}$

In [15, 12], several analogies between Eisenstein series and Eisenstein polynomials were reported. Suppose p is a prime number and v_p is the corresponding valuation for the field $\mathbb{Q}$. Then $a \in \mathbb{Q}$ is called p-integral if $v_p(a) \geq 0$. Eisenstein series have the following properties:

(1) All of the zeros of the Eisenstein series are on the circle $\{e^{\sqrt{-1}\theta} \mid \pi/2 \leq \theta \leq 2\pi/3\}$ [17].

(2) The zeros of the Eisenstein series $\psi_k^{\Gamma_1}(z)$ are the same as those for $\psi_{k+2}^{\Gamma_1}(z)$ [13].

(3) For odd prime p, where $p \geq 5$, the coefficients of the Eisenstein series $\psi_{p-1}^{\Gamma_1}(z)$ are p-integral [7, P. 233, Theorem 3], [9].

Oura's conjecture states that the analogous properties of (1), (2), and (3) also hold for $Th(\widetilde{\varphi}_\ell)$. Namely,

Conjecture 1.1 ([15, 12])**.** (1) *All of the zeros of $Th(\widetilde{\varphi_\ell^{G_{\mathrm{II}}}})$ are on the circle $\{e^{\sqrt{-1}\theta} \mid \pi/2 \leq \theta \leq 2\pi/3\}$.*

(2) *The zeros of $Th(\widetilde{\varphi_\ell^{G_{\mathrm{II}}}})$ are the same as those of $Th(\widetilde{\varphi_{\ell+4}^{G_{\mathrm{II}}}})$.*

(3) *Let p be an odd prime. The coefficients of $Th(\widetilde{\varphi_{2(p-1)}^{G_{\mathrm{II}}}})$ are p-integral.*

To explain our results, we introduce the zeta polynomials, which were defined by Duursma [2]. Analogous to coding theory, we say $f \in \mathbb{C}[x, y]$ is the formal weight enumerator of degree n if f is a homogeneous polynomial of degree n and the coefficient of x^n is one. Also, for

$$f(x, y) = x^n + \sum_{i=d}^{n} A_i x^{n-i} y^i \ (A_d \neq 0),$$

d is the minimum distance of f. Let R be a commutative ring and $R[[T]]$ be the formal power series ring over R. For $Z(T) = \sum_{i=0}^{\infty} a_n T_n \in R[[T]]$, $[T^k]Z(T)$ denotes the coefficient a_k. The following lemma follows:

Lemma 1.1 (cf. [2]). *Let f be a formal weight enumerator of degree n, d be the minimum distance, and q be any real number not one. Then there exists a unique polynomial $P_f(T) \in \mathbb{C}[T]$ of degree at most $n - d$ such that the following equation holds:*

$$[T^{n-d}] \frac{P_f(T)}{(1-T)(1-qT)} (xT + y(1-T))^n = \frac{f(x, y) - x^n}{q - 1}.$$

Definition 1.1 (cf. [3]). For a formal weight enumerator f, we call the polynomial $P_f(T)$ determined in Lemma 1.1 the zeta polynomial of f with respect to q. If all the zeros of $P_f(T)$ have absolute value $1/\sqrt{q}$, then we say that f satisfies the Riemann hypothesis analogues (RHA).

In [10], we investigated the zeta polynomials of the Eisenstein polynomials for Type II. In the following, we assume that $q = 2$. Below are the cases of $\ell = 8$ and $\ell = 12$:

Table 3: Examples of zeta polynomials

ℓ	$P_{\widetilde{\varphi_\ell^{G_{\mathrm{II}}}}}(T)$
8	$\frac{1}{5} + \frac{2T}{5} + \frac{2T^2}{5}$
12	$-\frac{1}{15} - \frac{2T}{15} - \frac{2T^2}{15} + \frac{4T^4}{15} + \frac{8T^5}{15} + \frac{8T^6}{15}$

In the previous paper, it was shown that Oura's observation for the zeta polynomial associated with Eisenstein polynomials holds:

Theorem 1.1 ([10]). (I) (1) $P_{\widetilde{\varphi_\ell^{G_{\mathrm{II}}}}}(T)$ *satisfies RHA.*

(2) *The zeros of* $P_{\widetilde{\varphi_\ell^{G_{\mathrm{II}}}}}(T)$ *interlace those of* $P_{\widetilde{\varphi_{\ell+4}^{G_{\mathrm{II}}}}}(T)$.

(3) *Let p be an odd prime with $p \neq 5$. Then the coefficients of* $P_{\widetilde{\varphi_{2(p-1)}^{G_{\mathrm{II}}}}}(T)$ *are p-integral.*

(II) *Let p be an odd prime. Then the coefficients of* $\widetilde{\varphi_{2(p-1)}^{G_{\mathrm{II}}}}(x,y)$ *are p-integral.*

(III) *Conjecture 1.1 (3) is true.*

The main porpose of the present paper is to show that the similar results hold for the remaining cases. We set the number w_{X} and q_{X} for a unified definition:

Type I: $w_{\mathrm{I}} = 2$, $q_{\mathrm{I}} = 2$,
Type II: $w_{\mathrm{II}} = 4$, $q_{\mathrm{II}} = 2$,
Type III: $w_{\mathrm{III}} = 3$, $q_{\mathrm{III}} = 3$,
Type IV: $w_{\mathrm{IV}} = 2$, $q_{\mathrm{IV}} = 4$.

Theorem 1.2. *For* $\mathrm{X} \in \{\mathrm{I}, \mathrm{III}, \mathrm{IV}\}$,

(I) (1) $P_{\widetilde{\varphi_\ell^{G_{\mathrm{X}}}}}(T)$ *satisfies RHA for* q_{X}.

(2) *The zeros of* $P_{\widetilde{\varphi_\ell^{G_{\mathrm{X}}}}}(T)$ *interlace those of* $P_{\widetilde{\varphi_{\ell+w_{\mathrm{X}}}^{H_1}}}(T)$.

(3) *Let p be an odd prime and assume that $p \neq 3$ for the case Type III. Then the coefficients of* $P_{\widetilde{\varphi_{2(p-1)}^{G_{\mathrm{X}}}}}(T)$ *are p-integral.*

(II) *Let p be an odd prime and assume that $p \neq 3$ for the case Type III. Then the coefficients of* $\widetilde{\varphi_{2(p-1)}^{G_{\mathrm{X}}}}(x,y)$ *are p-integral.*

(III) *Let p be an odd prime and assume that $p \neq 3$ for the case Type III. The coefficients of* $Th(\widetilde{\varphi_{2(p-1)}^{G_{\mathrm{X}}}})$ *are p-integral.*

In Section 2, the proof of Theorem 1.2 is provided along with concluding remarks.

2 Proof of Theorem 1.2

In this section, we provide the proof of Theorem 1.2.

2.1 Preliminaries

Before proving Theorem 1.2, we first recall a property of zeta polynomials.

The zeta polynomial $P_f(T)$ associated with f is related to the normalized weight enumerator of f as follows:

Definition 2.1 (cf. [4])**.** For a formal weight enumerator $f(x,y)=\sum_{i=0}^{n} A_i x^{n-i}y^i$, we define the normalized weight enumerator as follows:

$$N_f(t)=\frac{1}{q-1}\sum_{i=d}^{n} A_i/\binom{n}{i}t^{i-d}.$$

$P_f(T)$ and $N_f(t)$ have the following relation:

Theorem 2.1 (cf. [4])**.** *For a given formal weight enumerator $f(x,y)$ with minimum distance d, the zeta polynomial $P_f(T)$ and the normalized weight enumerator $N_f(t)$ have the following relation:*

$$\frac{P_f(T)}{(1-T)(1-qT)}(1-T)^{d+1}\equiv N_f\left(\frac{T}{1-T}\right) \pmod{T^{n-d+1}}.$$

2.2 The explicit form of the Eisenstein polynomials

The explicit form of the Eisenstein polynomials $\varphi_\ell^{G_{\mathrm{X}}}(x,y)$ are given by

Theorem 2.2. (1) Type I:

$$\begin{aligned}&\widetilde{\varphi_\ell^{G_{\mathrm{I}}}}(x,y)\\ &=\begin{cases} x^\ell+y^\ell+\dfrac{2}{2+\sqrt{2}^\ell}\displaystyle\sum_{0<j<\ell,\, j\equiv 0 \pmod 2}\binom{\ell}{j}x^{\ell-j}y^j & \textit{if } \ell\equiv 0 \pmod 2,\\ 0 & \textit{if } \ell\not\equiv 0 \pmod 2.\end{cases}\end{aligned}$$

(2) Type III:

$$\begin{aligned}&\widetilde{\varphi_\ell^{G_{\mathrm{III}}}}(x,y)\\ &=\begin{cases} x^\ell+\dfrac{3}{3+\sqrt{3}^\ell}\displaystyle\sum_{0<j<\ell,\, j\equiv 0 \pmod 3}2^j\binom{\ell}{j}x^{\ell-j}y^j & \textit{if } \ell\equiv 0 \pmod 4,\\ 0 & \textit{if } \ell\not\equiv 0 \pmod 4.\end{cases}\end{aligned}$$

(3) Type IV:

$$\widetilde{\varphi_\ell^{G_{\mathrm{IV}}}}(x,y) = \begin{cases} x^\ell + \dfrac{2}{2+2^\ell} \displaystyle\sum_{0<j<\ell, j\equiv 0 \pmod 2} 3^j \binom{\ell}{j} x^{\ell-j} y^j & \textit{if } \ell \equiv 0 \pmod 2, \\ 0 & \textit{if } \ell \not\equiv 0 \pmod 2. \end{cases}$$

Proof. We prove only for the case Type I. In Appendix A, we give the proof for the other cases.

By a direct calculation,

$$\begin{aligned} &\widetilde{\varphi_\ell^{G_{\mathrm{I}}}}(x,y) \\ &\quad = \frac{1}{2} \frac{1}{1+2(\frac{1}{\sqrt{2}})^\ell} \left(x^\ell + \left(\frac{1}{\sqrt{2}}(x+y)\right)^\ell + \left(\frac{1}{\sqrt{2}}(x-y)\right)^\ell + y^\ell \right. \\ &\quad \left. +(-x)^\ell + \left(-\frac{1}{\sqrt{2}}(x+y)\right)^\ell + \left(-\frac{1}{\sqrt{2}}(x-y)\right)^\ell + (-y)^\ell \right). \end{aligned}$$

Then the result follows.

Note that the elements of G_{I} are listed in one of the author's homepage [11].

□

2.3 The explicit form of the zeta polynomials

To prove Theorem 1.2, we provide the explicit formula of the zeta function associated with Eisenstein polynomials:

The zeta polynomial associated with Eisenstein polynomials $\widetilde{\varphi_\ell^{G_{\mathrm{X}}}}$ is written as follows:

Theorem 2.3. (1) Type I: *For* $\ell \equiv 0 \pmod 2$,

$$P_{\widetilde{\varphi_\ell^{G_{\mathrm{I}}}}}(T) = \frac{2+\sqrt{2}^\ell T^{\ell-2}}{2+\sqrt{2}^\ell}.$$

(2) Type III: *For* $\ell \equiv 0 \pmod 4$,

$$P_{\widetilde{\varphi_\ell^{G_{\mathrm{III}}}}}(T) = \frac{3\cdot 2^2}{3+\sqrt{3}^\ell} \sum_{j=0}^{(\ell-4)/2} (-3)^j T^{2j}.$$

(3) Type IV: *For* $\ell \equiv 0 \pmod 2$,

$$P_{\widetilde{\varphi_\ell^{G_{\mathrm{IV}}}}}(T) = \frac{6}{2+2^\ell}\sum_{j=0}^{\ell-2}(-2)^j T^j.$$

Proof. We prove only for the case Type I. In Appendix B, we give the proof for the other cases.

Let $N_{\widetilde{\varphi_\ell^{G_{\mathrm{I}}}}}$ be the normalized weight enumerator of $\varphi_\ell^{G_{\mathrm{I}}}$. By Definition 2.1, we have

$$\begin{aligned} N_{\widetilde{\varphi_\ell^{G_{\mathrm{I}}}}}(t) &= \sum_{0<j<\ell, j\equiv 0 \pmod 2} \frac{2}{2+\sqrt{2}^\ell} t^{j-2} + t^{\ell-2} \\ &\equiv \frac{2}{2+\sqrt{2}^\ell}\frac{1}{1-t^2} + \left(1 - \frac{2}{2+\sqrt{2}^\ell}\right) t^{\ell-2} \pmod{t^{\ell-1}}. \end{aligned}$$

Then, by Theorem 2.1, we have

$$\begin{aligned} &\frac{P_{\widetilde{\varphi_\ell^{G_{\mathrm{I}}}}}(T)}{(1-T)(1-2T)}(1-T)^3 \equiv N_{\widetilde{\varphi_\ell^{G_{\mathrm{I}}}}}\left(\frac{T}{1-T}\right) \pmod{T^{\ell-1}} \\ \Leftrightarrow & P_{\widetilde{\varphi_\ell^{G_{\mathrm{I}}}}}(T) \equiv N_{\widetilde{\varphi_\ell^{G_{\mathrm{I}}}}}\left(\frac{T}{1-T}\right)\frac{(1-T)(1-2T)}{(1-T)^3} \pmod{T^{\ell-1}} \\ &\equiv \frac{2}{2+\sqrt{2}^\ell}\frac{(1-T)^2}{(1-T)^2-T^2}\frac{(1-T)(1-2T)}{(1-T)^3} \\ &+ \left(1-\frac{2}{2+\sqrt{2}^\ell}\right)\left(\frac{T}{1-T}\right)^{\ell-2}\frac{(1-T)(1-2T)}{(1-T)^3} \pmod{T^{\ell-1}} \\ &\equiv \frac{2+\sqrt{2}^\ell T^{\ell-2}}{2+\sqrt{2}^\ell} \pmod{T^{\ell-1}}. \end{aligned}$$

Then, we have

$$P_{\widetilde{\varphi_\ell^{G_{\mathrm{I}}}}}(T) = \frac{2+\sqrt{2}^\ell T^{\ell-2}}{2+\sqrt{2}^\ell}.$$

□

2.4 Proof of Theorem 1.2

In this section, we provide the proof of Theorem 1.2.

First, the following lemma:

Lemma 2.1. *Let $\ell = 2(p-1)$ for some odd prime.*

(1) *We have*

$$2+\sqrt{2}^{\ell} \not\equiv 0 \pmod{p}.$$

(2) *Assume also that $p \neq 3$. Then we have*

$$3+\sqrt{3}^{\ell} \not\equiv 0 \pmod{p}.$$

(3) *We have*

$$2+2\ell \not\equiv 0 \pmod{p}.$$

Proof. We prove only for the case (1). The other cases can be proved similarly.

By Fermat's little theorem,

$$2+\sqrt{2}^{\ell} = 2+2^{p-1} \equiv 2 \not\equiv 0 \pmod{p}.$$

□

We now present the proof of Theorem 1.2.

Proof of Theorem 1.2. We prove only for the case Type I. The other cases can be proved similarly.

Clearly (I)–(1) and (I)–(2) follow from Theorem 2.3.

For (I)–(3), we recall that

$$P_{\widetilde{\varphi_{\ell}^{G_{\mathrm{I}}}}}(T) = \frac{2+\sqrt{2}^{\ell}T^{\ell-2}}{2+\sqrt{2}^{\ell}}.$$

Then, from Lemma 2.1 (1), the proof of Theorem 1.2 (I)–(3) is complete.

To show (II), we first recall that

$$\widetilde{\varphi_{\ell}^{G_{\mathrm{I}}}}(x,y) = x^{\ell}+y^{\ell}+\frac{2}{2+\sqrt{2}^{\ell}}\sum_{0<j<\ell, j\equiv 0 \pmod{2}} \binom{\ell}{j} x^{\ell-j}y^{j}.$$

By Lemma 2.1 (1), the coefficients of $\widetilde{\varphi_{2(p-1)}^{G_{\mathrm{I}}}}(x, y)$ are p-integral.

Finally, we show (III). By Theorem 1.2 (II), the coefficients of

$$\widetilde{\varphi_{\ell}^{G_{\mathrm{I}}}}(x, y) = x^{\ell} + y^{\ell} + \frac{2}{2+\sqrt{2}^{\ell}} \sum_{0<j<\ell, j\equiv 0 \pmod 2} \binom{\ell}{j} x^{\ell-j} y^{j}$$

are p-integral. The theta map f_0 and f_1 have integral Fourier coefficients. This completes the proof. □

2.5 Concluding Remarks

Remark 2.1. (1) The definition of the Eisenstein polynomial for genus g is given [14, 16]. In the present paper, we only consider the genus one ($g = 1$) case. For the cases with $g > 1$, do the analogies still hold?

(2) For $\mathrm{X} \in \{\mathrm{I}, \dots, \mathrm{IV}\}$, the group G_{X} is a finite unitary reflection group. These groups are classified in [18], which gives rise to a natural question: for the other unitary reflection groups, do our analogies still hold?

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A Proof of Theorem 2.2 for the cases Type III and Type IV

Proof of Theorem 2.2 (2). By a direct calculation,

$$
\begin{aligned}
&\widetilde{\varphi_\ell^{G_{\mathrm{III}}}}(x,y)\\
&\quad = \frac{1}{4}\frac{1}{1+3(\frac{1}{\sqrt{3}})^\ell}\\
&\quad \left(x^\ell + \left(\frac{1}{\sqrt{3}}(x+2y)\right)^\ell\right.\\
&\quad + \left(\frac{1}{\sqrt{3}}(x+\sqrt{-3}y-y)\right)^\ell + \left(\frac{1}{\sqrt{3}}(\sqrt{-1}x+\sqrt{3}y-\sqrt{-1}y)\right)^\ell\\
&\quad + (-x)^\ell + \left(-\frac{1}{\sqrt{3}}(x+2y)\right)^\ell\\
&\quad + \left(-\frac{1}{\sqrt{3}}(x+\sqrt{-3}y-y)\right)^\ell + \left(-\frac{1}{\sqrt{3}}(\sqrt{-1}x+\sqrt{3}y-\sqrt{-1}y)\right)^\ell\\
&\quad + (\sqrt{-1}x)^\ell + \left(\frac{\sqrt{-1}(x+2y)}{\sqrt{3}}\right)^\ell\\
&\quad + \left(\frac{1}{3}\sqrt{-1}\left(\sqrt{3}x-\left(\sqrt{3}-3\sqrt{-1}\right)y\right)\right)^\ell + \left(-\frac{x}{\sqrt{3}}+\sqrt{-1}y+\frac{y}{\sqrt{3}}\right)^\ell\\
&\quad + (-\sqrt{-1}x)^\ell + \left(-\frac{\sqrt{-1}(x+2y)}{\sqrt{3}}\right)^\ell\\
&\quad \left. + \left(\frac{x}{\sqrt{3}}-\frac{1}{3}\left(\sqrt{3}+3\sqrt{-1}\right)y\right)^\ell + \left(-\frac{\sqrt{-1}x}{\sqrt{3}}+y+\frac{\sqrt{-1}y}{\sqrt{3}}\right)^\ell\right).
\end{aligned}
$$

Then the result follows.

Note that the elements of G_{III} are listed in one of the author's homepage [11].

□

Proof of Theorem 2.2 (3). By a direct calculation,

$$
\begin{aligned}
&\widetilde{\varphi_\ell^{G_{\mathrm{IV}}}}(x,y)\\
&\quad = \frac{1}{2}\frac{1}{1+2(\frac{1}{2})^\ell}\left(x^\ell + \left(\frac{1}{2}(x+3y)\right)^\ell + \left(\frac{1}{2}(x-3y)\right)^\ell\right.\\
&\quad \left. +(-x)^\ell + \left(-\frac{1}{2}(x+3y)\right)^\ell + \left(-\frac{1}{2}(x-3y)\right)^\ell\right).
\end{aligned}
$$

Then the result follows.

Note that the elements of G_{IV} are listed in one of the author's homepage [11]. □

B Proof of Theorem 2.3 for the cases Type III and Type IV

Proof of Theorem 2.3 (2). Let $N_{\widetilde{\varphi_\ell^{G_{\mathrm{III}}}}}$ be the normalized weight enumerator of $\varphi_\ell^{G_{\mathrm{III}}}$. By Definition 2.1, we have

$$
\begin{aligned}
N_{\widetilde{\varphi_\ell^{G_{\mathrm{III}}}}}(t) &= \frac{1}{2}\frac{3}{3+3^{\ell/2}}\sum_{0<j<\ell, j\equiv 0 \pmod 3} 2^j t^{j-3}\\
&\equiv \frac{3\cdot 2^2}{3+3^{\ell/2}}\frac{1}{1-(2T)^3} \pmod{t^{\ell-2}}.
\end{aligned}
$$

Then, by Theorem 2.1, we have

$$
\begin{aligned}
&\frac{P_{\widetilde{\varphi_\ell^{G_{\mathrm{III}}}}}(T)}{(1-T)(1-3T)}(1-T)^4 \equiv N_{\widetilde{\varphi_\ell^{G_{\mathrm{III}}}}}\left(\frac{T}{1-T}\right) \pmod{T^{\ell-2}}\\
\Leftrightarrow & P_{\widetilde{\varphi_\ell^{G_{\mathrm{III}}}}}(T) \equiv N_{\widetilde{\varphi_\ell^{G_{\mathrm{III}}}}}\left(\frac{T}{1-T}\right)\frac{(1-T)(1-3T)}{(1-T)^4} \pmod{T^{\ell-2}}\\
&\equiv \frac{3\cdot 2^2}{3+3^{\ell/2}}\frac{(1-T)^3}{(1-T)^3-(2T)^3}\frac{(1-T)(1-3T)}{(1-T)^4} \pmod{T^{\ell-2}}\\
&\equiv \frac{3\cdot 2^2}{3+3^{\ell/2}}\frac{1}{1+3T^2} \pmod{T^{\ell-2}}.
\end{aligned}
$$

Then, we have

$$P_{\widetilde{\varphi_\ell^{G_{\mathrm{III}}}}}(T) = \frac{3\cdot 2^2}{3+3^{\ell/2}} \sum_{j=0}^{(\ell-4)/2} (-3)^j T^{2j}.$$

$\square$

Proof of Theorem 2.3 (3). Let $N_{\widetilde{\varphi_\ell^{G_{\mathrm{IV}}}}}$ be the normalized weight enumerator of $\varphi_\ell^{G_{\mathrm{IV}}}$. By Definition 2.1, we have

$$\begin{aligned} N_{\widetilde{\varphi_\ell^{G_{\mathrm{IV}}}}}(t) &= \frac{1}{3}\frac{2}{2+2^\ell} \sum_{0<j<\ell, j\equiv 0 \pmod 2} 3^j t^{j-2} \\ &\equiv \frac{2\cdot 3}{2+2^\ell}\frac{1}{1-(3T)^2} \pmod{t^{\ell-1}}. \end{aligned}$$

Then, by Theorem 2.1, we have

$$\begin{aligned} &\frac{P_{\widetilde{\varphi_\ell^{G_{\mathrm{IV}}}}}(T)}{(1-T)(1-4T)}(1-T)^3 \equiv N_{\widetilde{\varphi_\ell^{G_{\mathrm{IV}}}}}\left(\frac{T}{1-T}\right) \pmod{T^{\ell-1}} \\ \Leftrightarrow\, &P_{\widetilde{\varphi_\ell^{G_{\mathrm{IV}}}}}(T) \equiv N_{\widetilde{\varphi_\ell^{G_{\mathrm{IV}}}}}\left(\frac{T}{1-T}\right)\frac{(1-T)(1-4T)}{(1-T)^3} \pmod{T^{\ell-1}} \\ &\equiv \frac{1}{3}\frac{2}{2+2^\ell}\frac{(1-T)^2}{(1-T)^2-(3T)^2}\frac{(1-T)(1-4T)}{(1-T)^3} \pmod{T^{\ell-1}} \\ &\equiv \frac{6}{2+2\ell}\frac{1}{1+2T} \pmod{T^{\ell-1}}. \end{aligned}$$

Then, we have

$$P_{\widetilde{\varphi_\ell^{G_{\mathrm{IV}}}}}(T) = \frac{6}{2+2\ell}\sum_{j=0}^{\ell-2}(-2)^j T^j.$$

$\square$

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