MODULAR FORMS OF WEIGHT 8 FOR $\Gamma_g(1,2)$

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Abstract. We complete the program indicated by the Ansatz of D'Hoker and Phong in genus 4 by proving the uniqueness of the restriction to Jacobians of the weight 8 Siegel cusp forms satisfying the Anstaz. We prove $\dim[\Gamma_4(1,2),8]_0 = 2$ and $\dim[\Gamma_4(1,2),8] = 7$. In each genus, we classify the linear relations among the self-dual lattices of rank 16. We extend the program to genus 5 by constructing the unique linear combination of theta series that satisfies the Ansatz.

1. Introduction

Modular forms of weight 8 with respect to the theta group $\Gamma_g(1,2)$ have recently been a useful tool in Physics. Some are fundamental in the construction of a chiral superstring measure, cf. [6], [7], [8], [9] and [14], for small genera. Let $\mathcal{H}_g$ be the Siegel upper half space and let the *Jacobian locus*, $\mathrm{Jac}_g \subseteq \mathcal{H}_g$, be the set of period matrices of compact Riemann surfaces. For $g \leq 3$, Jac_g is dense in $\mathcal{H}_g$. According to the Anstaz of D'Hoker and Phong, we wish to find a modular form, $\Xi^{(g)}[0]$, of weight 8 with respect to $\Gamma_g(1,2)$ possessing the splitting property (1) and the vanishing trace property (2):

$$\Xi^{(g)}[0]\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix} = \Xi^{(g-k)}[0](\tau_1)\,\Xi^{(k)}[0](\tau_2) \tag{1}$$

with $\tau_1 \in \mathrm{Jac}_{g-k}$ and $\tau_2 \in \mathrm{Jac}_k$ and

$$\Xi^{(g)} = \mathrm{Tr}\left(\Xi^{(g)}[0]\right) = \sum_{\gamma\in\Gamma_g(1,2)\backslash\Gamma_g} \Xi^{(g)}[0]|_8\gamma = \sum_{\text{even chars } m} \Xi^{(g)}[m] \tag{2}$$

vanishes along Jac_g.

Due to the splitting property and the fact that the accepted solution in genus one, $\Xi^{(1)}[0](\tau) = \eta(\tau)^{12}\theta_0(\tau)^4$, is a cusp form, all of the $\Xi^{(g)}[0]$ should be cusp forms on the Jacobian locus. For $g \leq 4$, the $\Xi^{(g)}[0]$ will also be Siegel cusp forms. The existence of cusp forms with the properties (1) and (2) provides some vindication of the Anstaz of D'Hoker

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and Phong. Further vindication would be provided by the uniqueness of the restriction to the moduli space of curves, the relevant domain for physics. We formulate this as a third condition.

(3) Let $\Xi^{(1)}[0] = \eta^{12}\theta_0^4$. If $\Xi^{(g)}[0]$, $\tilde{\Xi}^{(g)}[0] \in [\Gamma_g(1,2), 8]$ both satisfy properties (1) and (2), then $\Xi^{(g)}[0] = \tilde{\Xi}^{(g)}[0]$ upon restriction to Jac_g.

The splitting property (1) means that the chiral superstring measure is the product measure on the block diagonal. The trace property (2) means that the cosmological constant vanishes. The uniqueness property (3) means that the desired family of solutions, $\{\Xi^{(g)}[0]\}$, is uniquely determined on the Jacobian locus by the genus one solution $\Xi^{(1)}[0]$.

D'Hoker and Phong gave an expression for $\Xi^{(2)}[0]$ and they proved that is well defined (uniqueness). In [2] the existence and the uniqueness of $\Xi^{(3)}[0]$ has been proven. The existence of $\Xi^{(4)}[0]$ has been proven in [14] and [5]. Moreover in [3] a relative uniqueness for $\Xi^{(4)}[0]$ has been proven. This relative uniqueness is due to two facts.

First in genus 4, there is a cusp form of weight 8 with respect to the full integral symplectic group Γ_4, the Schottky form J that vanishes along Jac_4 and hence along the block diagonal period matrices in genus 4. If $\Xi^{(4)}[0]$ satisfies properties (1) and (2), then so does $\Xi^{(4)}[0] + cJ$; thus the two desired properties cannot be uniquely satisfied by a Siegel modular cusp form in genus four. Accordingly, the uniqueness property (3) only requires the uniqueness of the restriction of $\Xi^{(g)}[0]$ to the moduli space of curves and this is consistent with the treatment in physics.

Second, in [3] the uniqueness of $\Xi^{(4)}[0]$ is proved only among the modular forms that are polynomials in the second order theta-constants. We know, as a consequence of the results in [21] and [22], that when $g \le 3$ all modular forms with respect to $\Gamma_g(1,2)$ are polynomials in the second order theta-constants. Recently in [18] two of the authors proved that this statement is false when $g \ge 4$.

Hence, a priori, in genus 4, there could be other modular forms with the properties (1) and (2) that cannot be written as polynomials in the second order theta-constants. Our first goal in this article is to prove the uniqueness property (3) in genus 4 and thus complete the program begun by D'Hoker and Phong through genus 4. The proof is based on the knowledge of the optimal slope of a cusp form in genus 4.

Our second goal is to construct $\Xi^{(5)}[0]$ in genus 5. In every genus, we classify the linear relations among the eight classes of self-dual lattices of rank 16. We use this knowledge to construct $\Xi^{(5)}[0]$ as a linear combination of theta series. We prove the relative uniqueness of $\Xi^{(5)}[0]$:

it is the only linear combination of theta series in genus 5 satisfying properties (1) and (2).

Using the methods of classical automorphic forms, we give independent proofs in all genera $g \leq 5$, although for motivation we are highly indebted to [3], [6], [7], [8], [9], [14] and [25]. To state our main Theorem, we introduce $J^{(g)} = \vartheta^{(g)}_{E_8 \oplus E_8} - \vartheta^{(g)}_{D_{16}^+}$ for arbitrary genus g. If $g = 4$, we get the Schottky form J which we already mentioned. We prove:

Theorem 1. *Let $\vartheta^{(g)}$ be the vector of the genus g theta series of the six odd self-dual rank 16 lattices. Let $\{\hat{\Xi}_j\}_{j=0}^5 \subseteq \mathbb{C}^6$ be the dual basis to $\{\tau^j\}_{j=0}^5 \subseteq \mathbb{C}^6$ for $\tau^j = (0, 2^j, 4^j, 8^j, 16^j, 32^j)$ and $\tau^0 = (1,1,1,1,1,1)$. Set $\hat{\Xi}_j^{(g)} = \hat{\Xi}_j \cdot \vartheta^{(g)}$. For all $g \geq 0$, we have*

$$\vartheta^{(g)} = \tau^5\, \hat{\Xi}_5^{(g)} + \tau^4\, \hat{\Xi}_4^{(g)} + \tau^3\, \hat{\Xi}_3^{(g)} + \tau^2\, \hat{\Xi}_2^{(g)} + \tau\, \hat{\Xi}_1^{(g)} + 1\, \hat{\Xi}_0^{(g)}. \tag{4}$$

Set $\Xi^{(g)}[0] = \hat{\Xi}_g^{(g)} - \frac{17 \cdot 89 \cdot 227}{2^{19} \cdot 3 \cdot 5 \cdot 7^2 \cdot 33} J^{(g)} \in [\Gamma_g(1,2), 8]$. For $g \leq 4$, the $\Xi^{(g)}[0]$ are cusp forms but $\Xi^{(5)}[0]$ is only a cusp form on the Jacobian locus. The family $\{\Xi^{(g)}[0]\}$ satisfies both properties (1) and (2) of the Ansatz for $g \leq 5$ and property (3) for $g \leq 4$. Also, $\Xi^{(5)}[0]$ is the unique linear combination of theta series in $[\Gamma_5(1,2), 8]$ that satisfies both properties (1) and (2).

And so we leave genus 5 where we found genus 4, with the construction of a Siegel modular form that satisfies Ansatz properties (1) and (2) but whose uniqueness is only proven among the span of the theta series. The uniqueness property (3) for $\Xi^{(5)}[0]$ would follow if we could prove $\dim[\Gamma_5(1,2), 8] = 8$.

2. Notation and known results

For a domain $\mathbb{D} \subseteq \mathbb{C}$, let $V_g(\mathbb{D})$ be the g-by-g symmetric matrices with coefficients in $\mathbb{D}$. For $\mathbb{D} \subseteq \mathbb{R}$, let $\mathcal{P}_g(\mathbb{D})^{\mathrm{semi}} \subseteq V_g(\mathbb{D})$ be the semidefinite elements and let $\mathcal{P}_g(\mathbb{D})$ be the definite elements. Let $\mathcal{H}_g$ be the Siegel upper half space of genus g, i.e. the set of g-by-g symmetric complex matrices with positive definite imaginary part. The symplectic group $\Gamma_g = \mathrm{Sp}(g, \mathbb{Z})$ acts on $\mathcal{H}_g$ via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ \tau := (A\tau + B)(C\tau + D)^{-1}.$$

Here we think of elements of Γ_g as consisting of four $g \times g$ blocks.

For $r \in \frac{1}{2}\mathbb{Z}$ and $\gamma \in \Gamma_g$, we set

$$(f|_r\gamma)(\tau) = \det(C\tau + D)^{-r} f(\gamma \circ \tau)$$

for some choice of square root. Let Γ be a subgroup of finite index in Γ_g, we say that a holomorphic function f defined on $\mathcal{H}_g$ is a modular form of weight r with respect to Γ if

$$\forall \gamma \in \Gamma, \forall \tau \in \mathcal{H}_g, \quad (f|_r\gamma)(\tau) = f(\tau).$$

and if additionally f is holomorphic at all cusps when $g = 1$. We denote by $[\Gamma, r]$ the vector space of such functions.

For holomorphic $f : \mathcal{H}_g \to \mathbb{C}$ we define

$$\Phi(f)(\tau_1) = \lim_{\lambda \longrightarrow +\infty} f \begin{pmatrix} \tau_1 & 0 \\ 0 & i\lambda \end{pmatrix}$$

when this limit exists for all $\tau_1 \in \mathcal{H}_{g-1}$. In particular, this operator maps $[\Gamma_g, r]$ to $[\Gamma_{g-1}, r]$ and $[\Gamma_g(1,2), r]$ to $[\Gamma_{g-1}(1,2), r]$. This operator has a relevant importance in the theory of modular forms, we refer to [16] or [12] for details. In the case of the full modular group, a cusp form is a modular form that is in the kernel of the Φ operator. In the case of a subgroup of the modular group, a modular form is a cusp form if

$$\forall \gamma \in \Gamma_g, \quad \Phi(f|_r\gamma) = 0.$$

We shall denote by $[\Gamma,\ k]_0$ the subspace of cusp forms. We shall also use the Witt homomorphism $\Psi_{ij}^* : [\Gamma_{i+j}(1,2), k] \to [\Gamma_i(1,2), k] \otimes [\Gamma_j(1,2), k]$ that is the pullback of the map $\Psi_{ij} : \mathcal{H}_i \times \mathcal{H}_j \to \mathcal{H}_{i+j}$ defined by $\Psi_{ij}(\tau_1, \tau_2) = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$. For all $f \in [\Gamma_g(1,2), k]$ we have the following formula for the Fourier coefficients of the image $\Psi_{ij}^* f$:

$$a\left(T_1 \otimes T_2; \Psi_{ij}^* f\right) = \sum_{T = \begin{pmatrix} T_1 & * \\ * & T_2 \end{pmatrix} \in \frac{1}{2}\mathcal{P}_g(\mathbb{Z})} a(T; f). \tag{5}$$

3. The Theta Group

Before proceeding to the theta group, we recall the theta functions. For $\tau \in \mathcal{H}_g$, $z \in \mathbb{C}^g$ and $\varepsilon,\ \delta \in \mathbb{F}_2^g$, where $\mathbb{F}_2$ denotes the abelian group $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, the associated theta function with characteristic $m = [\varepsilon, \delta]$ is

$$\begin{aligned} \theta_m(\tau, z) &= \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, z) \\ &= \sum_{n \in \mathbb{Z}^g} e\left(1/2 \cdot (n + \varepsilon/2)'\tau(n + \varepsilon/2) + (n + \varepsilon/2)'(z + \delta/2)\right). \end{aligned}$$

Here we denote by X' the transpose of X and $e(\ \ )$ stands for $\exp(2\pi i\ \ )$. As a function of z, $\theta_m(\tau, z)$ is odd or even depending on whether the scalar product $\varepsilon\cdot\delta \in \mathbb{F}_2$ is equal to 1 or 0, respectively. Theta-constants are restrictions of theta functions to $z = 0$. The product of two theta-constants is a modular form of weight 1 with respect to a subgroup of finite index of Γ_g.

Now, we discuss the basic properties of the theta group, $\Gamma_g(1,2)$. For a g-by-g real matrix X, we let $X_0 \in \mathbb{R}^g$ denote its diagonal. We write $\Gamma_g(n, 2n)$ for the subgroup of Γ_g defined by $\gamma \equiv 1_{2g} \bmod n$ and

$$(AB')_0 \equiv (CD')_0 \equiv 0 \bmod 2n.$$

Because the theta group is stable under transpose, we may also use the conditions $(A'C)_0 \equiv (B'D)_0 \equiv 0 \bmod 2n$.

The significance of the theta group $\Gamma_g(1,2)$ is the following: Γ_g acts on $\mathbb{F}_2^{2g}$ via

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}' \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} (A'C)_0 \\ (B'D)_0 \end{bmatrix}.$$

If we set $\zeta = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we may write this more compactly as $\zeta \cdot \gamma = \gamma'\zeta + \epsilon(\gamma)$, where $\epsilon(\gamma) = \begin{bmatrix} (A'C)_0 \\ (B'D)_0 \end{bmatrix}$. The existence of this action follows from the classical transformation of the theta-constants

$$\forall \gamma \in \Gamma_g, \quad \theta[\zeta]|_{1/2}\gamma \in e\left(\tfrac{1}{8}\mathbb{Z}\right) \theta[\zeta \cdot \gamma].$$

From the definitions it is not difficult to see that

$$\Gamma_g(1,2) = \mathrm{Stab}_{\Gamma_g} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^8 \in [\Gamma_g(1,2), 4].$$

The group Γ_g acts transitively on the even characteristics, so that the index of $\Gamma_g(1,2)$ in Γ_g is $2^{g-1}(2^g + 1)$ and we have the coset decomposition

$$\Gamma_g = \bigcup_{\text{even } \zeta} \Gamma_g(1,2)\,\gamma_\zeta,$$

where $\gamma_\zeta \in \Gamma_g$ is any element with $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdot \gamma_\zeta = \zeta$ in $\mathbb{F}_2^{2g}$. The stabilizers of the nonzero characteristics are given by the conjugate subgroups $\mathrm{Stab}_{\Gamma_g}[\zeta] = \gamma_\zeta^{-1}\Gamma_g(1,2)\gamma_\zeta$.

We let $\Delta_g(\mathbb{Z})$ denote the subgroup of Γ_g with "$C = 0$." The double coset decomposition of Γ_g with respect to $\Gamma_g(1,2)$ and $\Delta_g(\mathbb{Z})$ is of primary interest to us. The following Proposition shows that there are just two double cosets: the I-cusp, TIC, and the other cusp, TOC.

Proposition 2. *We have the double coset decomposition*

$$\Gamma_g = \Gamma_g(1,2)\Delta_g(\mathbb{Z}) \cup \Gamma_g(1,2)\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}\Delta_g(\mathbb{Z}),$$

where $I = I_g$ *denotes the identity matrix of degree* g*. The first double coset contains* 2^g *single cosets, the second* $2^{g-1}(2^g-1)$*.*

Proof. We prove the double coset decomposition by relating it to the single coset decomposition, so we first give representatives γ_ζ for the single coset decomposition. For even $\zeta = \begin{bmatrix} a \\ b \end{bmatrix}$, we may choose γ_ζ as

$$\gamma_\zeta = \begin{pmatrix} I & 0 \\ \mathrm{diag}(a) & I \end{pmatrix}\begin{pmatrix} I & S \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & S \\ \mathrm{diag}(a) & I + \mathrm{diag}(a)S \end{pmatrix}$$

with any $S \in V_g(\mathbb{Z})$ such that $Sa + S_0 = b$. When $a = 0$, we just take $S = \mathrm{diag}(b)$. When $a \neq 0$, one way to get S is to take $S = \beta b' + b\beta'$ for any *odd* characteristic $\begin{bmatrix} a \\ \beta \end{bmatrix}$.

We will show that the right action of $\Delta_g(\mathbb{Z})$ on the set of the even characteristics has two orbits: $\begin{bmatrix} 0 \\ * \end{bmatrix}$ with 2^g elements and $\begin{bmatrix} \neq 0 \\ * \end{bmatrix}$ with $2^{g-1}(2^g-1)$ elements. For $\zeta = \begin{bmatrix} 0 \\ b \end{bmatrix}$, we have $\gamma_\zeta \in \Gamma_g(1,2)\Delta_g(\mathbb{Z})$ because $\gamma_\zeta = \begin{pmatrix} I & \mathrm{diag}(b) \\ 0 & I \end{pmatrix}$. For $\zeta = \begin{bmatrix} a \\ b \end{bmatrix}$, with $a \neq 0$, we have $\gamma_\zeta \in \Gamma_g(1,2)\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}\Delta_g(\mathbb{Z})$ because for any $U \in \mathrm{GL}_g(\mathbb{Z})$ we have

$$\begin{aligned}\gamma_\zeta &= \begin{pmatrix} I & 0 \\ \mathrm{diag}(a) & I \end{pmatrix}\begin{pmatrix} I & \beta b' + b\beta' \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} U^{-1} & 0 \\ \mathrm{diag}(a)U^{-1} - U' & U' \end{pmatrix}\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & (U')^{-1} \end{pmatrix}\begin{pmatrix} I & \beta b' + b\beta' \\ 0 & I \end{pmatrix}.\end{aligned}$$

To complete the proof we show that $\begin{pmatrix} U^{-1} & 0 \\ \mathrm{diag}(a)U^{-1} - U' & U' \end{pmatrix} \in \Gamma_g(1,2)$ for some choice of U. Choose U so that $U'I_0 \equiv a \mod 2$. The defining conditions, $(AB')_0 \equiv (CD')_0 \equiv 0 \bmod 2$, are then satisfied if we note that $(U'U)_0 \equiv U'I_0 \mod 2$. □

The choice of $\gamma_\zeta = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$ and of $\zeta = \begin{bmatrix} I_0 \\ 0 \end{bmatrix}$ as a representative for the nontrivial double coset is somewhat arbitrary. We have chosen the even characteristic $\begin{bmatrix} I_0 \\ 0 \end{bmatrix}$ because its upper and lower parts are invariant under permutations and the representative γ_ζ because it is a direct sum of $\mathrm{SL}_2(\mathbb{Z})$ matrices.

We now study the Fourier expansions at these two cusps. For $S \in V_g(\mathbb{R})$, let $t(S) = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{R})$. For $U \in \mathrm{GL}_g(\mathbb{R})$, let $u(U) = \begin{pmatrix} U & 0 \\ 0 & (U')^{-1} \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{R})$. For $A, B \in V_g(\mathbb{C})$, set $\langle A, B \rangle = \mathrm{tr}(AB)$.

The I-Cusp. The translation lattice for the I-cusp (TIC) is

$$\{S \in V_g(\mathbb{Z}) : t(S) \in \Gamma_g(1,2)\} = \{S \in V_g(\mathbb{Z}) : \ S \text{ even}\}.$$

The dual lattice with respect to $\langle \ , \ \rangle$ is $\frac{1}{2}V_g(\mathbb{Z})$. If we multiply the above translation lattice by $1/2$, we get the lattice $\mathcal{X}_g$ consisting of "half-integral" matrices, which is nothing else but the dual lattice of $V_g(\mathbb{Z})$. The similarity lattice for TIC is

$$\{U \in \mathrm{GL}_g(\mathbb{Z}) : u(U) \in \Gamma_g(1,2)\} = \mathrm{GL}_g(\mathbb{Z}).$$

Therefore, an $f \in [\Gamma_g(1,2), k]_0$ has a Fourier expansion at TIC

$$f(\tau) = \sum_T a(T) e\left(\langle \tau, T \rangle\right)$$

where $T \in \frac{1}{2}\mathcal{P}_g(\mathbb{Z})$ runs over integral forms multiplied by $\frac{1}{2}$ and, for all $U \in \mathrm{GL}_g(\mathbb{Z})$, $a(U'TU) = \det(U)^k \, a(T)$.

The Other Cusp. Let $M = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$ and let $\Gamma_g(1,2)^M$ denote $M^{-1}\Gamma_g(1,2)M$. The translation lattice $\mathcal{L}$ for the other cusp (TOC) is given by S such that $t(S)$ stabilizes $\begin{bmatrix} I_0 \\ 0 \end{bmatrix}$.

$$\begin{aligned} \mathcal{L} &= \{S \in V_g(\mathbb{Z}) : t(S) \in \Gamma_g(1,2)^M\} \\ &= \{S \in V_g(\mathbb{Z}) : \ SI_0 + S_0 \equiv 0 \mod 2\}. \end{aligned}$$

Lemma 3. *Let $E_{ij} \in V_g(\mathbb{Z})$ be the matrix with 1 in the (i,j) and (j,i) spots and zeroes elsewhere. The lattice $\mathcal{L}$ contains $2V_g(\mathbb{Z})$, $\mathrm{diag}(\mathbb{Z}^g)$ and $E_{ij} + E_{jk} + E_{ki}$ for all distinct triples $i,j,k \in \{1,2,\ldots,g\}$.*

Proof. This is an easy computation from the defining condition: $SI_0 + S_0 \equiv 0 \mod 2$. □

Actually, the above elements span $\mathcal{L}$, although the last group $E_{ij} + E_{jk} + E_{ki}$ is linearly dependent over $\mathbb{F}_2$ for $g > 3$. Furthermore, the indices are $[V_g(\mathbb{Z}) : \mathcal{L}] = 2^{g-1}$ and $[\mathcal{L} : 2V_g(\mathbb{Z})] = 2^{1+\binom{g}{2}}$ but we need none of these facts.

Definition 4. *Elements of the lattice* $4\mathcal{L}^*$ *are called* **very even**.

From $2V_g(\mathbb{Z}) \subseteq \mathcal{L} \subseteq V_g(\mathbb{Z})$ we see that the dual lattice, $\mathcal{L}^*$, satisfies $\frac{1}{2}\mathcal{X}_g \supseteq \mathcal{L}^* \supseteq \mathcal{X}_g$, so that the elements of $4\mathcal{L}^*$ are in fact even. Since $\operatorname{diag}(\mathbb{Z}^g) \subseteq \mathcal{L}$, the diagonals of $\mathcal{L}^*$ are integral and the diagonals of very even forms are multiples of 4.

The similarity group $\mathcal{G}$ for TOC is given by the U such that $u(U)$ stabilizes $\begin{bmatrix} I_0 \\ 0 \end{bmatrix}$:

$$\begin{aligned} \mathcal{G} &= \{U \in \mathrm{GL}_g(\mathbb{Z}) : u(U) \in \Gamma_g(1,2)^M\} \\ &= \{U \in \mathrm{GL}_g(\mathbb{Z}) : \ U' I_0 \equiv I_0 \mod 2 \ \}. \end{aligned}$$

A useful comment here is that $\mathcal{G}$ contains all permutation matrices and all diagonal sign changes.

Therefore, an $f \in [\Gamma_g(1,2), k]_0$ has a Fourier expansion at TOC

$$(f|M)(\tau) = \sum_T b(T) e\left(\langle \tau, T \rangle\right)$$

where $T \in \mathcal{L}^* \cap \mathcal{P}_g(\mathbb{Q})$ runs over very even forms multiplied by $\frac{1}{4}$ and, for all $U \in \mathcal{G}$, $b(U'TU) = \det(U)^k \, b(T)$.

Proposition 5. *Let* $f \in [\Gamma_g(1,2), k]$. *The form* f *is a cusp form if and only if* $\Phi(f) = 0$ *and* $\Phi(f|M) = 0$.

Furthermore, let the coset representatives for $\Gamma_g(1,2)\backslash\Gamma_g$ *be written*

$$\gamma_\zeta = \begin{pmatrix} I & 0 \\ \operatorname{diag}(a) & I \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} = M_a \, t(S).$$

For $S \in V_g(\mathbb{Z})$, *let* $\pi(S) \in V_{g-1}(\mathbb{Z})$ *be the lower* $(g-1)$*-by-*$(g-1)$ *block.*
For $a = (0; a')$, *we have* $\Phi(f|M_a \, t(S)) = \Phi(f)|M_{a'} \, t(\pi(S))$.
For $a = I_0 + (0; c')$, *we have* $\Phi(f|M_a \, t(S)) = \Phi(f|M)|M_{c'} \, t(\pi(S))$.

Proof. The form f is a cusp form if and only if $\Phi(f|\gamma_\zeta) = 0$ for every γ_ζ in the single coset decomposition of $\Gamma_g(1,2)\backslash\Gamma_g$. We rely on two properties of the Φ map. First, for any $F \in [\Gamma, k]$, we have $\Phi(F|t(S)) = \Phi(F)|t(\pi(S))$. Second, let $I_2 \in \Gamma_1$ be the identity matrix; for $\gamma' \in \Gamma_{g-1}$, we have $\Phi(F|I_2 \oplus \gamma') = \Phi(F)|\gamma'$. Here we understand that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & A & 0 & B \\ c & 0 & d & 0 \\ 0 & C & 0 & D \end{pmatrix}.$$

There are two cases depending upon a. If a begins with 0, so that $a = (0; a')$, then $M_a = I_2 \oplus M_{a'}$ and

$$\Phi(f|M_a\, t(S)) = \\ \Phi(f|M_a)|t(\pi(S)) = \Phi(f|I_2 \oplus M_{a'})|t(\pi(S)) = \Phi(f)|M_{a'}\, t(\pi(S)).$$

If a begins with 0, define $c = a - I_0$; then c begins with 0, so that $c = (0; c')$. Note that $M = M_{I_0}$ and that $M_a = MM_c$. We have

$$\Phi(f|M_a\, t(S)) = \Phi(f|MM_c)|t(\pi(S)) = \Phi(f|M)|M_{c'}\, t(\pi(S)).$$

Thus, the cases $\Phi(f|\gamma_\zeta) = 0$ follow from $\Phi(f) = 0$ and $\Phi(f|M) = 0$. □

4. Lemmata

Let $\Gamma \subseteq \Gamma_g$ be a subgroup of finite index $I = [\Gamma_g : \Gamma]$. The Norm map is defined as

$$\begin{aligned} \mathrm{Norm} : [\Gamma, k] &\to [\Gamma_g, Ik] \\ f &\mapsto \prod_{\gamma \in \Gamma \backslash \Gamma_g} f|_k \gamma. \end{aligned}$$

This map naturally induces the map between the associated projective spaces and we use the same notation Norm again. The next Lemma shows that if we have a nontrivial subspace S of Siegel cusp forms, all of whose elements have a norm that is a multiple of a fixed form, then the dimension of S is one.

Lemma 6. *Let $\Gamma \subseteq \Gamma_g$ be a subgroup of finite index I. Let $S \subseteq [\Gamma, k]_0$ be a subspace. If* $\mathrm{Norm} : \mathbb{P}(S) \to \mathbb{P}\left([\Gamma_g, Ik]_0\right)$ *has an image consisting of precisely one point, then* $\dim S = 1$.

Proof. Take $A, B \in S \setminus \{0\}$. We will show that B/A is constant as a meromorphic function on $\mathcal{H}_g$, and thus conclude that $\dim S = 1$. Let $\xi = \mathrm{Norm}(A) \in [\Gamma_g, Ik]_0$ and note that $\xi \neq 0$ since $A \neq 0$. By assumption, for all $x \in \mathbb{C}$ there exists a $c \in \mathbb{C}$ such that $\mathrm{Norm}(Ax + B) = c\,\xi$. We will show that c is a polynomial in x. We can evaluate c by picking $\tau_0 \in \mathcal{H}_g$ with $\xi(\tau_0) \neq 0$; then

$$c = \frac{\mathrm{Norm}(Ax + B)(\tau_0)}{\xi(\tau_0)} = \prod_{\gamma}^{I} \left(x + \frac{(B|\gamma)(\tau_0)}{(A|\gamma)(\tau_0)} \right).$$

However, since $c = \mathrm{Norm}(Ax+B)/\xi$ as well, we have

$$\prod_{\gamma}^{I}\left(x+\frac{(B|\gamma)}{(A|\gamma)}\right)=\prod_{\gamma}^{I}\left(x+\frac{(B|\gamma)(\tau_0)}{(A|\gamma)(\tau_0)}\right).$$

Letting $x=-B/A$, we see that B/A is a meromorphic function with a discrete image and hence is a constant. □

Definition 7. *A function $\phi : \mathcal{P}_g(\mathbb{R})^{semi} \to \mathbb{R}_{\geq 0}$ is called type one if*
1. For all $s \in \mathcal{P}_g(\mathbb{R})$, $\phi(s) > 0$,
2. for all $\lambda \in \mathbb{R}_{\geq 0}$ and $s \in \mathcal{P}_g(\mathbb{R})^{semi}$, $\phi(\lambda s) = \lambda\phi(s)$,
3. for all $s_1, s_2 \in \mathcal{P}_g(\mathbb{R})^{semi}$, $\phi(s_1+s_2) \geq \phi(s_1)+\phi(s_2)$.

Type one functions are continuous on $\mathcal{P}_g(\mathbb{R})$ and respect the partial order on $\mathcal{P}_g(\mathbb{R})^{\mathrm{semi}}$. We will need the acquaintance of two type one functions: the Minimum function $m(s) = \inf_{u\in\mathbb{Z}^g\setminus\{0\}} u'su$ and its convex dual $w(s) = \inf_{u\in\mathcal{P}_g(\mathbb{R})} \langle u, s\rangle/m(u)$, the dyadic trace.

Recall the definition of the *slope*.

$$\text{For } f \in [\Gamma,k]_0, \text{ let } \mathrm{supp}(f) = \{T \in \mathcal{P}_g(\mathbb{Q}) : a(T;f) \neq 0\};$$
$$m(f) = \min m\left(\mathrm{supp}(f)\right) \text{ and } \mathrm{slope}(f) = \frac{k}{m(f)}.$$

We know that the minimal slope on $\mathcal{A}_g = \Gamma_g\backslash\mathcal{H}_g$ is: 12, 10, 9, 8 for $g = 1,2,3,4$, respectively, cf. [24] and in particular the Corollary to Theorem 3. It is rather easy to check that any Siegel modular form that attains this slope is a power of $\Delta \in [\Gamma_1, 12]_0$, $X_{10} \in [\Gamma_2, 10]_0$, $X_{18} \in [\Gamma_3, 18]_0$ or $J \in [\Gamma_4, 8]_0$ in $g = 1,2,3$ or 4, respectively.

In the next Lemma, we extend this result to subgroups Γ of index I by looking at the slope of the average vanishing, $k/\mu(f)$, where

$$\mu(f) = \frac{1}{I}\sum_{\gamma\in\Gamma\backslash\Gamma_g}^{I} m\left(f|\gamma\right).$$

Lemma 8. *Let $\Gamma \subseteq \Gamma_g$ be a subgroup of finite index I. Let $f \in [\Gamma,k]_0$. If $k/\mu(f)$ is the optimal value:* 12, 10, 9, 8 *for $g = 1,2,3,4$, respectively, then* $\mathrm{Norm}(f) \in [\Gamma_g, Ik]_0$ *is a constant multiple of a power of Δ, X_{10}, X_{18}, J, respectively.*

Proof. The slope of the level one $\mathrm{Norm}(f)$ is $Ik/m\left(\mathrm{Norm}(f)\right)$. Therefore, it suffices to prove that $m\left(\mathrm{Norm}(f)\right) \geq I\mu(f)$. Note that m

is a type one function satisfying $m(s_1+s_2) \geq m(s_1)+m(s_2)$. We have

$$\begin{aligned} m\left(\mathrm{Norm}(f)\right) &= m\left(\prod_{\gamma\in\Gamma\backslash\Gamma_g} f|\gamma\right) = \min m\left(\mathrm{supp}\left(\prod_{\gamma}^{I} f|\gamma\right)\right) \\ &\geq \min m\left(\sum_{\gamma}^{I}\mathrm{supp}(f|\gamma)\right) \geq \min \sum_{\gamma}^{I} m\left(\mathrm{supp}(f|\gamma)\right) \\ &= \sum_{\gamma}^{I} \min m(\mathrm{supp}(f|\gamma)) = \sum_{\gamma}^{I} m(f|\gamma) = I\mu(f). \quad \square \end{aligned}$$

The dyadic trace, defined as $w(s) = \inf_{u\in\mathcal{P}_g(\mathbb{R})} \langle u,s\rangle/m(u)$, also has a characterization as a supremum [19]. A *dyadic representation* of a form $T \in \mathcal{P}_g(\mathbb{Q})$ is given by $\alpha_i > 0$ and $v_i \in \mathbb{Z}^g\backslash\{0\}$ such that $T = \sum_i \alpha_i v_i v_i'$. Since $\langle T,u\rangle = \sum_i \alpha_i \langle v_i v_i', u\rangle \geq \sum_i \alpha_i m(u)$, it follows from the definition of the dyadic trace that $w(T) \geq \sum_i \alpha_i$ for any dyadic representation. In fact, we have

$$w(T) = \sup\{\sum_i \alpha_i : \text{ over all dyadic representations } \sum_i \alpha_i v_i v_i' \text{ of } T\}$$

and that this supremum is attained by a particular dyadic representation. The dyadic trace is a useful tool in the geometry of numbers.

Lemma 9. *Let $T \in \mathcal{P}_g(\mathbb{Q})$ with $g = g_1+g_2$, $T = \begin{pmatrix} T_1 & W \\ W' & T_2\end{pmatrix}$ for $T_1 \in \mathcal{P}_{g_1}(\mathbb{Q})$, $T_2 \in \mathcal{P}_{g_2}(\mathbb{Q})$, $W \in \mathrm{Mat}_{g_1\times g_2}(\mathbb{Q})$. We have*

$$w(T) \leq w(T_1) + w(T_2).$$

Furthermore, we have equality if and only if $W = 0$.

Proof. Let $T = \sum_i \alpha_i v_i v_i'$ with $\alpha_i > 0$ and $v_i \in \mathbb{Z}^g \setminus \{0\}$ be a dyadic representation of T that attains the dyadic trace: $w(T) = \sum_i \alpha_i$. We use $\pi_1(v)$, $\pi_2(v)$ to denote the first g_1 and the last g_2 coordinates of v. For $j = 1, 2$

$$T_j = \sum_{i:\pi_j(v_i)\neq 0} \alpha_i \pi_j(v_i)\pi_j(v_i)'$$

is a dyadic representation of T_j so that $w(T_j) \geq \sum_{i:\pi_j(v_i)\neq 0} \alpha_i$. Therefore we have

$$w(T_1) + w(T_2) \geq \sum_i \alpha_i + \sum_{i:\pi_1(v_i)\neq 0 \text{ and } \pi_2(v_i)\neq 0} \alpha_i \geq \sum_i \alpha_i = w(T).$$

This is the first assertion. Equality can be attained only if the second sum above is empty. In this case we have

$$\begin{aligned} T &= \sum_i \alpha_i v_i v_i' = \sum_i \alpha_i \begin{pmatrix} \pi_1 v_i \\ \pi_2 v_i \end{pmatrix} \begin{pmatrix} \pi_1 v_i' & \pi_2 v_i' \end{pmatrix} \\ &= \sum_{i:\pi_1(v_i)=0} \alpha_i \begin{pmatrix} \pi_1 v_i \\ \pi_2 v_i \end{pmatrix} \begin{pmatrix} \pi_1 v_i' & \pi_2 v_i' \end{pmatrix} + \sum_{i:\pi_2(v_i)=0} \alpha_i \begin{pmatrix} \pi_1 v_i \\ \pi_2 v_i \end{pmatrix} \begin{pmatrix} \pi_1 v_i' & \pi_2 v_i' \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & T_2 \end{pmatrix} + \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}. \end{aligned}$$

This shows $W = 0$ and so $w(T) = w(T_1) + w(T_2)$ indeed holds. □

Lemma 10. *Let $f \in [\Gamma_g(1,2), k]$ for $g \geq 2$. If $f \in \ker \Psi^*_{1,g-1}$, then we have $m(f) \geq 1$.*

Proof. Recall that the Fourier expansion of f is

$$f(\tau) = \sum_T a(\tfrac{1}{2}T; f) e\left(\langle \tau, \tfrac{1}{2}T \rangle\right)$$

with $T \in \mathcal{P}_g(\mathbb{Z})$ since $\ker \Psi^*_{1,g-1} \subseteq [\Gamma_g(1,2), k]_0$. For $\frac{1}{2}T \in \operatorname{supp}(f)$ we will show that $m(T) \geq 2$. It suffices to prove that $a\left(\frac{1}{2}(1 \oplus T_0); f\right) = 0$ for all $T_0 \in \mathcal{P}_{g-1}(\mathbb{Z})$ because, if $m(T) = 1$, then T is $\mathrm{GL}_g(\mathbb{Z})$-equivalent to $1 \oplus T_0$.

We will show that $a\left(\frac{1}{2}(1 \oplus T_0); f\right) = 0$ by induction on $w(T_0)$. The base case of the induction is satisfied because f is a cusp form. Since $\Psi^*_{1,g-1} f = 0$, its $\frac{1}{2}1 \otimes \frac{1}{2}T_0$ Fourier coefficient is also 0 and

$$0 = \sum_{v \in \mathbb{Z}^{g-1}} a\left(\tfrac{1}{2}\begin{pmatrix} 1 & v \\ v' & T_0 \end{pmatrix}\right) = a\left(\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & T_0 \end{pmatrix}\right) + \sum_{v \neq 0} a\left(\tfrac{1}{2}\begin{pmatrix} 1 & v \\ v' & T_0 \end{pmatrix}\right) \tag{6}$$

All the indices $\begin{pmatrix} 1 & v \\ v' & T_0 \end{pmatrix}$ are $\mathrm{GL}_g(\mathbb{Z})$-equivalent to $1 \oplus T_v$ for some $T_v \in \mathcal{P}_{g-1}(\mathbb{Z})$. By Lemma 9 we have

$$1 + w(T_v) = w\begin{pmatrix} 1 & 0 \\ 0 & T_v \end{pmatrix} = w\begin{pmatrix} 1 & v \\ v' & T_0 \end{pmatrix} < 1 + w(T_0) \text{ for } v \neq 0,$$

so that $w(T_v) < w(T_0)$ for $v \neq 0$. By the induction hypothesis, we have

$$a\left(\tfrac{1}{2}\begin{pmatrix} 1 & v \\ v' & T_0 \end{pmatrix}\right) = a\left(\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & T_v \end{pmatrix}\right) a\left(\tfrac{1}{2}(1 \oplus T_v); f\right) = 0$$

for $v \neq 0$ and so $a\left(\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & T_0 \end{pmatrix}\right) = 0$ by equation (6) as well. □

5. Dimensions

This next Theorem is a consequence of the work of Igusa and the facts previously discussed in genus three, cf. [22], [2].

Theorem 11. *For* $g = 1, 2, 3$, *we have* $\dim\left[\Gamma_g(1,2), 8\right]_0 = 1$.

Proof. Recall that these spaces are nonempty, containing nonzero elements $\vartheta^{(g)}_{1_8, P_4}$ (See the appendix for this function). The Minimum function m is a $\mathrm{GL}_g(\mathbb{Z})$-class function, so that $m(f|\gamma)$ only depends upon the double coset $\Gamma_g(1,2)\gamma\Delta_g(\mathbb{Z})$. Hence, for nontrivial $f \in \left[\Gamma_g(1,2), 8\right]_0$ we have, by the double coset decomposition of Proposition 2, (set $I = 2^{g-1}(2^g+1)$)

$$\mu(f) = \frac{1}{I}\sum_{\gamma\in\Gamma_g(1,2)\backslash\Gamma_g}^{I} m\left(f|\gamma\right) = \frac{2}{2^g+1}\,m(f) + \frac{2^g-1}{2^g+1}\,m(f|\begin{pmatrix} I & 0\\ I & I\end{pmatrix}).$$

The indices at TIC consist of $\frac{1}{2}$ times integral forms and so $m(f) \geq \frac{1}{2}$. The indices at TOC consist of $\frac{1}{4}$ times very even forms and so $m(f|\begin{pmatrix} I & 0\\ I & I\end{pmatrix}) \geq 1$. Therefore we have

$$\mu(f) \geq \frac{2}{2^g+1}\left(\frac{1}{2}\right) + \frac{2^g-1}{2^g+1}\,(1) = \frac{2^g}{2^g+1}$$

and we can make the following table of the maximum slope of the average vanishing for nontrivial elements of $\left[\Gamma_g(1,2), 8\right]_0$:

Table 1. Maximum slope for average vanishing.

g	$k/\mu(f)$
1	$\frac{8}{2/3} = 12$
2	$\frac{8}{4/5} = 10$
3	$\frac{8}{8/9} = 9.$

By Lemma 8, we know that $\mathrm{Norm}(f)$ is some multiple of $\Delta^2 \in [\Gamma_1, 24]$, $X_{10}^8 \in [\Gamma_2, 80]$, $X_{18}^{16} \in [\Gamma_3, 288]$, respectively.

The image of $\mathrm{Norm} : \mathbb{P}\left(\left[\Gamma_g(1,2), 8\right]_0\right) \to \mathbb{P}\left(\left[\Gamma_g, *\right]_0\right)$ consists of one point in these cases so that $\dim\left[\Gamma_g(1,2), 8\right]_0 = 1$ by Lemma 6. $\square$

Theorem 12. *For* $g = 2, 3$, *we have* $[\Gamma_g(1,2), 8] \cap \ker \Psi^*_{1,g-1} = \{0\}$. *For* $g = 4$, *we have* $[\Gamma_4(1,2), 8] \cap \ker \Psi^*_{1,3} = \mathbb{C}J$.

Proof. By Lemma 10, an $f \in [\Gamma_g(1,2), 8] \cap \ker \Psi^*_{1,g-1}$ has $m(f) \geq 1$. Therefore, as in the proof of the previous Theorem,

$$\mu(f) \geq \frac{2}{2^g+1}\,(1) + \frac{2^g-1}{2^g+1}\,(1) = 1$$

and f has $8/\mu(f)$ at most 8. Hence $f = 0$ in $g = 2, 3$ and $\mathrm{Norm}(f)$ is a multiple of J^{136} in $g = 4$ by Lemma 8.

In $g = 4$ therefore, the image of the map

$$\mathrm{Norm} : \mathbb{P}\left([\Gamma_4(1,2), 8] \cap \ker \Psi^*_{1,3}\right) \to \mathbb{P}\left([\Gamma_4, 8 \cdot 136]_0\right)$$

has just one point. Therefore, we have $\dim [\Gamma_4(1,2), 8] \cap \ker \Psi^*_{1,3} = 1$ by Lemma 6, and necessarily $[\Gamma_4(1,2), 8] \cap \ker \Psi^*_{1,3} = \mathbb{C}J$. □

6. Linear Relations among Theta series

A more general way to construct modular forms is to use theta series, in particular if L is a self-dual lattice of rank m, with 8 dividing m, then we have the associated quadratic form S and the theta series

$$\vartheta_L^{(g)}(\tau) = \sum_{X \in \mathbb{Z}^{m,\,g}} e(1/2 \cdot \mathrm{tr}(S[X]\tau))$$

is a modular form of weight $m/2$ relative to $\Gamma_g(1,2)$. We let $[\Gamma_g(1,2), k]^\vartheta$ denote the subspace spanned by theta series of self-dual lattices of rank $2k$.

There are eight self-dual lattices of rank 16, two even lattices and six odd. The theta series are elements of $[\Gamma_g, 8]$ and $[\Gamma_g(1,2), 8]$, respectively. In this section we find all the linear relations among these theta series for every genus. We give two applications. First, we derive the results $\dim[\Gamma_4(1,2), 8] = 7$ and $\dim[\Gamma_4(1,2), 8]_0 = 2$. Second, we push the physicists' program to success in genus five and prove our main Theorem 1.

The eight self-dual lattices of rank 16 are all found by Kneser's gluing method. We use the notation in [4], Table 16.7. For $4|n$, $D_n^+ = D_n \cup ([1] + D_n)$ is unimodular; D_8^+ is commonly denoted by E_8. The two even lattices are given by $E_8 \oplus E_8$ and D_{16}^+. An odd lattice is given by

$\mathbb{Z}^4 \oplus D_{12}^+$ and another by

$$(D_8 \oplus D_8)^+ = D_8 \oplus D_8 \cup ([1] \times [2] + D_8 \oplus D_8) \cup ([2] \times [1] + D_8 \oplus D_8) \cup ([3] \times [3] + D_8 \oplus D_8), \text{ where}$$

$$[1] \times [2] = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 0, 0, 0, 0, 0, 0, 0, 1],$$

$$[2] \times [1] = [0, 0, 0, 0, 0, 0, 0, 1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}],$$

$$[3] \times [3] = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}].$$

Also, we have $\mathbb{Z} \oplus A_{15}^+$ for

$$A_{15}^+ = A_{15} \cup ([4] + A_{15}) \cup ([8] + A_{15}) \cup ([12] + A_{15}),$$

where $[i]$, for $i + j = n + 1$, means j coordinates of $\frac{i}{n+1}$ followed by i coordinates of $\frac{-j}{n+1}$. Finally, we have $\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+$ for

$$(E_7 \oplus E_7)^+ = (E_7 \oplus E_7) \cup ([1] \times [1] + (E_7 \oplus E_7)), \text{ where}$$

$$[1] \times [1] = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}, -\frac{3}{4}; \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}, -\frac{3}{4}].$$

The following Table gives basic information about the theta series $\vartheta_i^{(g)} = \vartheta_{\Lambda_i}^{(g)}$ of these lattices, labeled as Λ_i for $i = 0, 1, \ldots, 7$. Here, τ_i is the number of vectors of norm one. It is the coefficient of $q^{1/2}$ in the genus one Fourier expansion $\vartheta_i^{(1)}$, whose leading term is also given. The space $[\Gamma_1(1,2), 8]$ is spanned by $\Xi^{(1)}[0]$, $\vartheta_0^{(1)}$ and $\vartheta_6^{(1)}$ and the coefficients τ_i, b_i, c_i, of the linear relation $\vartheta_i^{(1)} = \tau_i\, \Xi^{(1)}[0] + b_i\, \vartheta_0^{(1)} + c_i\, \vartheta_6^{(1)}$ are also given.

Table 2. The eight self-dual lattices of rank 16.

i	Λ_i	τ_i	b_i	c_i	$\vartheta_i^{(1)} - 1$
0	$(D_8 \oplus D_8)^+$	0	1	0	$224q^1 + 4096q^{3/2}$
1	$\mathbb{Z} \oplus A_{15}^+$	2	1	0	$2q^{1/2} + 240q^1 + 4120q^{3/2}$
2	$\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+$	4	1	0	$4q^{1/2} + 256q^1 + 4144q^{3/2}$
3	$\mathbb{Z}^4 \oplus D_{12}^+$	8	1	0	$8q^{1/2} + 288q^1 + 4192q^{3/2}$
4	$\mathbb{Z}^8 \oplus E_8$	16	1	0	$16q^{1/2} + 352q^1 + 4288q^{3/2}$
5	$\mathbb{Z}^{16}$	32	1	0	$32q^{1/2} + 480q^1 + 4480q^{3/2}$
6	$E_8 \oplus E_8$	0	0	1	$480q^1$
7	D_{16}^+	0	0	1	$480q^1$

We recall the general set up of [11] for organizing linear relations among theta series. Let $\{L_i\}_{i=1}^h$ be a set of self-dual lattices of rank $2k$. Let $V = \mathbb{C}^h$ and define $\Theta^{(g)} : V \to [\Gamma_g(1,2), k]$ by $\Theta^{(g)}(r) = \sum_{i=1}^h r_i \vartheta_{L_i}^{(g)}$ and the convention $\vartheta_{L_i}^{(0)} = 1$. We have a decreasing filtration $V_g = \ker \Theta^{(g)}$ of V and the dual increasing filtration $W_g = (V_g)^\perp$ of the dual space V^*. Composing the canonical isomorphism $\Theta^{(g)}(V) \cong V/V_g$ with the noncanonical $V/V_g \cong W_g$, we have $\dim[\Gamma_g(1,2), k]^\vartheta = \dim W_g$.

The Fourier coefficients of the theta series provide natural elements of W_g. For $T \in \frac{1}{2}\mathcal{P}_g(\mathbb{Z})$ and $\vartheta_{L_i}^{(g)}(\tau) = \sum_T a_i^{(g)}(T) e(\langle T, \tau\rangle)$, we define $w_g(T) \in V^*$ by $w_g(T)_i = a_i^{(g)}(T)$. The $w_g(T)$ for $T \in \frac{1}{2}\mathcal{P}_g(\mathbb{Z})$ span W_g and linear relations may be presented by giving a basis of W_g in terms of the $w_g(T)$ or linear combinations thereof. We also define the component-wise multiplication on V^* because this multiplication respects the W_g-filtration: $W_i W_j \subseteq W_{i+j}$. This follows from equation (5) but one should also note its more detailed consequence:

$$W_i W_j = \left(\ker \Psi_{ij}^* \circ \Theta^{(g)}\right)^\perp \subseteq \left(\ker \Theta^{(g)}\right)^\perp = W_{i+j}.$$

For example, the even self-dual lattices of rank 16, $\{E_8 \oplus E_8, D_{16}^+\}$, give the *problem of Witt*: find the dependence of the theta series in each genus. From results of Igusa and Kneser, cast in the above form, we have $W_0 = W_1 = W_2 = W_3 = \langle 1\rangle$ and $W_4 = V^*$ where 1 is the vector of all ones. This is a nice way to present the linear relations. By a result of Igusa [17], $J = \vartheta_6^{(4)} - \vartheta_7^{(4)}$ gives the Schottky form in genus 4. The representation numbers for D_4 follow from: $r(D_\ell, D_4) = 1152\binom{\ell}{4}$, $r(A_\ell, D_4) = 0$, $r(E_8, D_4) = 1152 \cdot 3150$, $r(E_7, D_4) = 1152 \cdot 315$, see [11].

Theorem 13. *For $V = \mathbb{C}^8$, let $\Theta^{(g)} : V \to [\Gamma_g(1,2), 8]$ be defined by $\Theta^{(g)}(r) = \sum_{i=0}^7 r_i \vartheta_i^{(g)}$ for the eight self-dual lattices of rank* 16. *For $c, \tau, \sigma \in V^*$ given by*

$$\begin{aligned}
&\tau = w_1(\tfrac{1}{2}) = (0, 2, 4, 8, 16, 32, 0, 0),\\
&\sigma = w_4\left(\tfrac{1}{2}D_4\right) = 1152\,(140, 0, 630, 496, 3220, 1820, 6300, 1820),\\
&c = (0, 0, 0, 0, 0, 0, 1, 1),
\end{aligned}$$

the filtration $W_g = \left(\ker \Theta^{(g)}\right)^\perp$ is given by $W_0 = \langle 1\rangle$, $W_1 = \langle 1, c, \tau\rangle$, $W_2 = \langle 1, c, \tau, \tau^2\rangle$, $W_3 = \langle 1, c, \tau, \tau^2, \tau^3\rangle$, $W_4 = \langle 1, c, \tau, \tau^2, \tau^3, \tau^4, \sigma\rangle$ and $W_5 = V^*$. The relation among the theta series in $g = 4$ is $\det(\vartheta^{(4)}, \sigma, \tau^4, \tau^3, \tau^2, \tau, c, 1) = 0$. *For the six odd lattices alone, the corresponding filtration is $W_0 = \langle 1\rangle$, $W_1 = \langle 1, \tau\rangle$, $W_2 = \langle 1, \tau, \tau^2\rangle$, $W_3 = \langle 1, \tau, \tau^2, \tau^3\rangle$ and $W_4 = V^*$.*

Proof. From the definition of $\Theta^{(0)}$, we see that $W_0 = \langle 1\rangle$. From Table 2, we see that $\vartheta^{(1)} = \tau\,\Xi^{(1)}[0] + (1-c)\vartheta_0^{(1)} + c\,\vartheta_6^{(1)}$ so that $W_1 = \langle 1, c, \tau\rangle$. By Theorem 12, the forms vanishing on the reducible locus $\mathcal{H}_1 \times \mathcal{H}_1$ are trivial, so that $W_2 = W_1W_1 = \langle 1, c, \tau, \tau^2\rangle$. By Theorem 12, the forms vanishing on the reducible locus $\mathcal{H}_1 \times \mathcal{H}_2$ are trivial, so that $W_3 = W_1W_2 = \langle 1, c, \tau, \tau^2, \tau^3\rangle$. Let $r \in V$ and note $W_1W_3 = \langle 1, c, \tau, \tau^2, \tau^3, \tau^4\rangle$. For $r \perp W_1W_3$, $\Theta^{(4)}(r)$ vanishes on the reducible locus $\mathcal{H}_1 \times \mathcal{H}_3$ and is hence a multiple of $J^{(4)}$ by Theorem 12. Thus $\Theta^{(4)}(r) = r \cdot \sigma J^{(4)}$ by looking at the Fourier coefficient for $\frac{1}{2}D_4$; therefore $W_4 = \langle 1, c, \tau, \tau^2, \tau^3, \tau^4, \sigma\rangle$ and the relation in $g = 4$ follows immediately. We have $W_5 \supseteq W_1W_4 = \langle 1, c, \tau, \tau^2, \tau^3, \tau^4, \sigma, c\sigma, \tau\sigma\rangle = V^*$. The corresponding result for the six odd lattices follows by restriction to the first six coordinates. □

Remark 14. *Hence,* $\dim[\Gamma_g(1,2), 8]^\vartheta$ *is* $3, 5, 6, 7, 8$ *for* $g = 1, 2, 3, 4, 5$.

It will be important to compute Witt images of bases for $[\Gamma_g(1,2), 8]^\vartheta$. For brevity, let

$$c_0 = \frac{1}{5160960}\,\frac{\det(\sigma, \tau^4, \tau^3, \tau^2, \tau, 1)}{\det(\tau^5, \tau^4, \tau^3, \tau^2, \tau, 1)} = \frac{89 \cdot 227}{2^{19} \cdot 3 \cdot 5 \cdot 7^2}.$$

Proposition 15. *Let* $\{\hat{\Xi}_j\}_{j=0}^5 \subseteq \mathbb{C}^6$ *be the dual basis to* $\{\tau^j\}_{j=0}^5 \subseteq \mathbb{C}^6$. *Write* $\hat{\Xi}_j^{(g)} = \Theta^{(g)}(\hat{\Xi}_j)$. *For* $g \le 4$, *we have* $\hat{\Xi}_g^{(g)} \in [\Gamma_g(1,2), 8]_0$. *We have* $\hat{\Xi}_0^{(g)} = \vartheta_0^{(g)}$ *and* $\hat{\Xi}_5^{(4)} = c_0 J^{(4)}$. *We have the Witt images*

$$\Psi_{14}^*\hat{\Xi}_5^{(5)} = \hat{\Xi}_1^{(1)} \otimes \hat{\Xi}_4^{(4)} + \left(62\,\hat{\Xi}_1^{(1)} + \vartheta_0^{(1)}\right) \otimes c_0 J^{(4)};\ \Psi_{23}^*\hat{\Xi}_5^{(5)} = \hat{\Xi}_2^{(2)} \otimes \hat{\Xi}_3^{(3)},$$
$$\Psi_{13}^*\hat{\Xi}_4^{(4)} = \hat{\Xi}_1^{(1)} \otimes \hat{\Xi}_3^{(3)};\quad \Psi_{22}^*\hat{\Xi}_4^{(4)} = \hat{\Xi}_2^{(2)} \otimes \hat{\Xi}_2^{(2)},$$
$$\Psi_{13}^*\hat{\Xi}_3^{(4)} = \hat{\Xi}_1^{(1)} \otimes \hat{\Xi}_2^{(3)} + \vartheta_0^{(1)} \otimes \hat{\Xi}_3^{(3)},$$
$$\Psi_{13}^*\hat{\Xi}_2^{(4)} = \hat{\Xi}_1^{(1)} \otimes \hat{\Xi}_1^{(3)} + \vartheta_0^{(1)} \otimes \hat{\Xi}_2^{(3)},$$
$$\Psi_{13}^*\hat{\Xi}_1^{(4)} = \hat{\Xi}_1^{(1)} \otimes \vartheta_0^{(3)} + \vartheta_0^{(1)} \otimes \hat{\Xi}_1^{(3)}.$$

Proof. Consider the filtration of Theorem 13 for the six odd lattices. For $g \le 4$, we have $\hat{\Xi}_g^{(g)} \in [\Gamma_g(1,2), 8]_0$ because $\hat{\Xi}_g$ is annihilated by $\langle 1, \tau, \ldots, \tau^{g-1}\rangle = W_{g-1}$. The relation $\hat{\Xi}_5^{(4)} = c_0 J^{(4)}$ follows from the $g = 4$ relation in Theorem 13 but we may also argue directly: $\hat{\Xi}_5$ is annihilated by $\langle 1, \tau, \ldots, \tau^4\rangle = W_1W_3$ and so $\hat{\Xi}_5^{(4)}$ is a form vanishing on the reducible locus $\mathcal{H}_1 \times \mathcal{H}_3$, necessarily $\hat{\Xi}_5^{(4)} = c\,J^{(4)}$ for some constant c by Theorem 12. By Cramer's rule we have

$$\hat{\Xi}_5^{(4)} = \det(\vartheta^{(4)}, \tau^4, \tau^3, \tau^2, \tau, 1)/\det(\tau^5, \tau^4, \tau^3, \tau^2, \tau, 1) = c\,J^{(4)}.$$

Evaluating at the Fourier coefficient for $\frac{1}{2}D_4$, we have $5160960c = \det(\sigma, \tau^4, \ldots, 1)/\det(\tau^5, \tau^4, \ldots, 1)$ so that $c = c_0$. The identity $\hat{\Xi}_0^{(g)} = \vartheta_0^{(g)}$ follows from $\tau_0 = 0$.

We now consider the Witt images. Write the map $\Theta^{(4)} \in V^* \otimes [\Gamma_g(1,2), 8]$ in the basis $\{\tau^j\}_{j=0}^5$ so that $\vartheta^{(4)} = \sum_j \Theta^{(4)}(\hat{\Xi}_j)\tau^j$ or

$$\vartheta^{(4)} = \tau^5\, c_0 J^{(4)} + \tau^4\, \hat{\Xi}_4^{(4)} + \tau^3\, \hat{\Xi}_3^{(4)} + \tau^2\, \hat{\Xi}_2^{(4)} + \tau\, \hat{\Xi}_1^{(4)} + 1\, \vartheta_0^{(4)}.$$

That the Witt images of Ψ_{13}^* are as stated follows from

$$\begin{aligned}
&\tau^4\, \Psi_{13}^* \hat{\Xi}_4^{(4)} + \tau^3\, \Psi_{13}^* \hat{\Xi}_3^{(4)} + \tau^2\, \Psi_{13}^* \hat{\Xi}_2^{(4)} + \tau\, \Psi_{13}^* \hat{\Xi}_1^{(4)} + 1\, \Psi_{13}^* \vartheta_0^{(4)} \\
=&\vartheta^{(1)} \otimes \vartheta^{(3)} \\
=&\left(\tau\, \hat{\Xi}_1^{(1)} + 1\, \vartheta_0^{(1)}\right) \otimes \left(\tau^3\, \hat{\Xi}_3^{(3)} + \tau^2\, \hat{\Xi}_2^{(3)} + \tau\, \hat{\Xi}_1^{(3)} + 1\, \vartheta_0^{(3)}\right) \\
=&\tau^4 \left(\hat{\Xi}_1^{(1)} \otimes \hat{\Xi}_3^{(3)}\right) + \tau^3 \left(\hat{\Xi}_1^{(1)} \otimes \hat{\Xi}_2^{(3)} + \vartheta_0^{(1)} \otimes \hat{\Xi}_3^{(3)}\right) + \\
&\tau^2 \left(\hat{\Xi}_1^{(1)} \otimes \hat{\Xi}_1^{(3)} + \vartheta_0^{(1)} \otimes \hat{\Xi}_2^{(3)}\right) + \tau \left(\hat{\Xi}_1^{(1)} \otimes \vartheta_0^{(3)} + \vartheta_0^{(1)} \otimes \hat{\Xi}_1^{(3)}\right) + 1 \vartheta_0^{(1)} \otimes \vartheta_0^{(3)}.
\end{aligned}$$

The others are similar but one needs to use $\tau^6 = 62\tau^5 - 1240\tau^4 + 9920\tau^3 - 31744\tau^2 + 32768\tau$. □

The splitting of these forms may be used to provide finer information.

Theorem 16. *We have* $[\Gamma_4(1,2), 8]_0 = \mathbb{C}J + \mathbb{C}\,\Xi^{(4)}[0]$.

Proof. Take $f \in [\Gamma_4(1,2), 8]_0$ and let $\Psi_{1,3}^* f = \alpha\, \Xi^{(1)}[0] \otimes \Xi^{(3)}[0]$. So $f - \alpha \hat{\Xi}_4^{(4)}[0]$ is in $\ker \Psi_{1,3}^*$ and is a multiple of J by Theorem 12. □

We wish to compute the dimension of $[\Gamma_g(1,2), 8]$ for $g \leq 4$. We know that $\dim [\Gamma_1(1,2), 8] = 3$, spanned by $\vartheta_0^{(1)}$, $\Xi^{(1)}[0]$ and $\vartheta_6^{(1)}$. Our method for $g = 3, 4$ does not succeed in $g = 2$, so we must make use of the result of Igusa and Runge that $\dim [\Gamma_2(1,2), 8] = 4$. A basis for $[\Gamma_2(1,2), 8]$ is then given by $\vartheta_0^{(2)}$, $\hat{\Xi}_1^{(2)}$, $\Xi^{(2)}[0]$ and $\vartheta_6^{(2)}$.

Theorem 17. *We have* $[\Gamma_3(1,2), 8] = [\Gamma_3(1,2), 8]^{\vartheta}$ *and the dimension is* 5. *We have* $[\Gamma_4(1,2), 8] = [\Gamma_4(1,2), 8]^{\vartheta}$ *and the dimension is* 7.

Proof. We omit the proof of this theorem for $g = 3$, since the fact is known and the method that we use is illustrated by the $g = 4$ case. We just observe that a basis of $[\Gamma_3(1,2), 8]^{\vartheta}$ is given by $\vartheta_0^{(3)}$, $\hat{\Xi}_1^{(3)}$, $\hat{\Xi}_2^{(3)}$, $\Xi^{(3)}[0]$ and $\vartheta_6^{(3)}$. We make use of the commutative diagram:

$$\begin{array}{ccc}
[\Gamma_4(1,2), k] & \xrightarrow{\Psi_{13}^*} & [\Gamma_1(1,2), k] \otimes [\Gamma_3(1,2), k] \\
\Psi_{112}^* \downarrow & & \mathrm{Id} \oplus \Psi_{12}^* \downarrow \\
\mathrm{Sym}\left([\Gamma_1(1,2), k]^{\otimes 2}\right) \otimes [\Gamma_2(1,2), k] & \rightarrow & \left([\Gamma_1(1,2), k]^{\otimes 2}\right) \otimes [\Gamma_2(1,2), k]
\end{array}$$

A basis of $[\Gamma_4(1,2),8]^\vartheta$ is given by $\vartheta_0^{(4)}$, $\hat{\Xi}_1^{(4)}$, $\hat{\Xi}_2^{(4)}$, $\hat{\Xi}_3^{(4)}$, $\hat{\Xi}_4^{(4)}$, $\hat{\Xi}_5^{(4)}$ and $\vartheta_6^{(4)}$. From Proposition 15, the images of Ψ_{13}^* are $\vartheta_0^{(1)} \otimes \vartheta_0^{(3)}$, $\Xi^{(1)}[0] \otimes \vartheta_0^{(3)} + \vartheta_0^{(1)} \otimes \hat{\Xi}_1^{(3)}$, $\Xi^{(1)}[0] \otimes \hat{\Xi}_1^{(3)} + \vartheta_0^{(1)} \otimes \hat{\Xi}_2^{(3)}$, $\Xi^{(1)}[0] \otimes \hat{\Xi}_2^{(3)} + \vartheta_0^{(1)} \otimes \Xi^{(3)}[0]$, $\Xi^{(1)}[0] \otimes \Xi^{(3)}[0]$, 0 and $\vartheta_6^{(1)} \otimes \vartheta_6^{(3)}$. These linearly dependent images span a 6 dimensional space inside $[\Gamma_1(1,2),8] \otimes [\Gamma_3(1,2),8]$. This shows that $\Psi_{13}^*[\Gamma_4(1,2),8]^\vartheta$ is 6 dimensional.

On the other hand, the general element of the 15 dimensional space $[\Gamma_1(1,2),8] \otimes [\Gamma_3(1,2),8]$ is given by

$$\begin{aligned}
&\alpha_1 \vartheta_0^{(1)} \otimes \vartheta_0^{(3)} + \alpha_2 \Xi^{(1)}[0] \otimes \vartheta_0^{(3)} + \alpha_3 \vartheta_6^{(1)} \otimes \vartheta_0^{(3)} + \\
&\beta_1 \vartheta_0^{(1)} \otimes \hat{\Xi}_1^{(3)} + \beta_2 \Xi^{(1)}[0] \otimes \hat{\Xi}_1^{(3)} + \beta_3 \vartheta_6^{(1)} \otimes \hat{\Xi}_1^{(3)} + \\
&\gamma_1 \vartheta_0^{(1)} \otimes \hat{\Xi}_2^{(3)} + \gamma_2 \Xi^{(1)}[0] \otimes \hat{\Xi}_2^{(3)} + \gamma_3 \vartheta_6^{(1)} \otimes \hat{\Xi}_2^{(3)} + \\
&\delta_1 \vartheta_0^{(1)} \otimes \Xi^{(3)}[0] + \delta_2 \Xi^{(1)}[0] \otimes \Xi^{(3)}[0] + \delta_3 \vartheta_6^{(1)} \otimes \Xi^{(3)}[0] + \\
&\epsilon_1 \vartheta_0^{(1)} \otimes \vartheta_6^{(3)} + \epsilon_2 \Xi^{(1)}[0] \otimes \vartheta_6^{(3)} + \epsilon_3 \vartheta_6^{(1)} \otimes \vartheta_6^{(3)}.
\end{aligned}$$

By Proposition 15, the image of this element under $\mathrm{Id} \oplus \Psi_{12}^*$ is

$$\begin{aligned}
&\left(\alpha_1 \vartheta_0^{(1)} + \alpha_2 \Xi^{(1)}[0] + \alpha_3 \vartheta_6^{(1)}\right) \otimes \vartheta_0^{(1)} \otimes \vartheta_0^{(2)} + \\
&\left(\beta_1 \vartheta_0^{(1)} + \beta_2 \Xi^{(1)}[0] + \beta_3 \vartheta_6^{(1)}\right) \otimes \left(\Xi^{(1)}[0] \otimes \vartheta_0^{(2)} + \vartheta_0^{(1)} \otimes \hat{\Xi}_1^{(2)}\right) + \\
&\left(\gamma_1 \vartheta_0^{(1)} + \gamma_2 \Xi^{(1)}[0] + \gamma_3 \vartheta_6^{(1)}\right) \otimes \left(\Xi^{(1)}[0] \otimes \hat{\Xi}_1^{(2)} + \vartheta_0^{(1)} \otimes \Xi^{(2)}[0]\right) + \\
&\left(\delta_1 \vartheta_0^{(1)} + \delta_2 \Xi^{(1)}[0] + \delta_3 \vartheta_6^{(1)}\right) \otimes \Xi^{(1)}[0] \otimes \Xi^{(2)}[0] + \\
&\left(\epsilon_1 \vartheta_0^{(1)} + \epsilon_2 \Xi^{(1)}[0] + \epsilon_3 \vartheta_6^{(1)}\right) \otimes \vartheta_6^{(1)} \otimes \vartheta_6^{(2)}.
\end{aligned}$$

If we demand that this image lie in $\mathrm{Sym}\left([\Gamma_1(1,2),8]^{\otimes 2}\right) \otimes [\Gamma_2(1,2),8]$, it imposes certain linear equations on the coefficients. Again, every term is a tensor of basis elements. The free parameters are α_1, δ_2 and ϵ_3. We have $\alpha_2 = \beta_1$, $\beta_2 = \gamma_1$ and $\gamma_2 = \delta_1$. We have $\alpha_3 = \beta_3 = \gamma_3 = \delta_3 = \epsilon_1 = \epsilon_2 = 0$. Thus the preimage $X = (\mathrm{Id} \oplus \Psi_{12}^*)^{-1}\left(\mathrm{Sym}\left([\Gamma_1(1,2),8]^{\otimes 2}\right) \otimes [\Gamma_2(1,2),8]\right)$ is 6 dimensional inside $[\Gamma_1(1,2),8] \otimes [\Gamma_3(1,2),8]$. This preimage X necessarily contains $\Psi_{13}^*[\Gamma_4(1,2),8]$. However, since $\Psi_{13}^*[\Gamma_4(1,2),8]^\vartheta$ is 6 dimensional we also have $\Psi_{13}^*[\Gamma_4(1,2),8]^\vartheta = X = \Psi_{13}^*[\Gamma_4(1,2),8]$.

From $\Psi_{13}^*[\Gamma_4(1,2),8]^\vartheta = \Psi_{13}^*[\Gamma_4(1,2),8]$ and the knowledge of the cusp forms, we can easily deduce $[\Gamma_4(1,2),8] = [\Gamma_4(1,2),8]^\vartheta$. For each $f \in [\Gamma_4(1,2),8]$, there is a $g \in [\Gamma_4(1,2),8]^\vartheta$ with $\Psi_{13}^* f = \Psi_{13}^* g$. We have

$\Psi_{13}^*(f-g)=0$ so that $f-g\in[\Gamma_4(1,2),8]_0$. Thus $f=g+\alpha\,\Xi^{(4)}[0]+\beta J^{(4)}\in[\Gamma_4(1,2),8]^{\vartheta}$. $\square$

Lemma 18. *Let $f\in[\Gamma_g(1,2),k]$ be such that $\Phi(f),\Phi(f|M)\in[\Gamma_{g-1},k]$. Then $\Phi\left(\mathrm{Tr}(f)\right)=2^{g-1}(1+2^{g-1})\,\Phi(f)+2^{2g-2}\,\Phi(f|M)$.*

Proof. The trace of f is given by $\mathrm{Tr}(f)=\sum_{\text{even }\zeta}f|\gamma_\zeta$. According to Proposition 5, when $\zeta=\begin{bmatrix}a\\b\end{bmatrix}$ and the first entry of a is zero, we have $\Phi(f|\gamma_\zeta)=\Phi(f|M_a\,t(S))=\Phi(f)|M_{a'}\,t(\pi(S))$. If $\Phi(f)$ is level one then $\Phi(f|\gamma_\zeta)=\Phi(f)$. There are $2^{g-1}(1+2^{g-1})$ even characteristics with the first entry of a zero.

When the first entry of a is one, we have $\Phi(f|\gamma_\zeta)=\Phi(f|M_a\,t(S))=\Phi(f|M)|M_{c'}\,t(\pi(S))$. If $\Phi(f|M)$ is level one then $\Phi(f|\gamma_\zeta)=\Phi(f|M)$. There are 2^{2g-2} even characteristics where the first entry of a is one. Thus, $\Phi\left(\mathrm{Tr}(f)\right)=2^{g-1}(1+2^{g-1})\,\Phi(f)+2^{2g-2}\,\Phi(f|M)$. $\square$

Corollary 19. *If $\hat{\Xi}_5^{(4)}=c_0J^{(4)}$, then $\mathrm{Tr}\,\hat{\Xi}_5^{(5)}=16\cdot17c_0J^{(5)}$.*

Proof. We know that $\Phi(\hat{\Xi}_5^{(5)})=\hat{\Xi}_5^{(4)}=c_0J^{(4)}$ is level one. We will show that $\Phi(\hat{\Xi}_5^{(5)}|M)=0$. By Proposition 15, we have $\Psi_{14}^*\hat{\Xi}_5^{(5)}=\hat{\Xi}_1^{(1)}\otimes\hat{\Xi}_4^{(4)}+(62\,\hat{\Xi}_1^{(1)}+\vartheta_0^{(1)})\otimes c_0J^{(4)}$. Using that $\hat{\Xi}_1^{(1)}$ is a cusp form, we have $\Phi(\hat{\Xi}_5^{(5)}|M)=\Phi(\vartheta_0^{(1)}|M)c_0J^{(4)}|M=0$ since $\vartheta_0^{(1)}$ vanishes at TOC. By Lemma 18, we have

$$\Phi(\mathrm{Tr}\,\hat{\Xi}_5^{(5)})=2^{g-1}(1+2^{g-1})\,\hat{\Xi}_5^{(4)}=16\cdot17c_0J^{(4)}.$$

So $\mathrm{Tr}\,\hat{\Xi}_5^{(5)}\in[\Gamma_5,8]$ has Φ image $16\cdot17c_0J^{(4)}$. Therefore, since [20], page 216, tells us that the only cusp forms in $[\Gamma_5,8]$ are trivial, we have $\mathrm{Tr}\,\hat{\Xi}_5^{(5)}=16\cdot17c_0J^{(5)}$. $\square$

Definition 20. *Let $f\in[\Gamma_g(1,2),k]$. We say that f is a cusp form on the Jacobian locus if for all $\gamma\in\Gamma_g$, $\Phi(f|\gamma)$ vanishes upon restriction to the period matrices of compact Riemann surfaces.*

We are ready for the *Proof of Theorem 1.* The map $\Theta^{(g)}:V\to[\Gamma_g(1,2),8]$ is written as $\vartheta^{(g)}$ in the standard basis and as $\sum_j\hat{\Xi}_j^{(g)}\tau^j$ in the $\{\tau^j\}$ basis so that we have equation 4. We set

$$\Xi^{(g)}[0]=\hat{\Xi}_g^{(g)}-\frac{17\cdot89\cdot227}{2^{19}\cdot3\cdot5\cdot7^2\cdot33}J^{(g)}\in[\Gamma_g(1,2),8].$$

Since the form $J^{(g)}$ vanishes along $\mathrm{Jac}_k\times\mathrm{Jac}_{g-k}$ when $g\le5$, as an immediate consequence of Proposition 15, we get the splitting property for $\hat{\Xi}_g^{(g)}$ and hence for $\Xi^{(g)}[0]$ along $\mathrm{Jac}_k\times\mathrm{Jac}_{g-k}$. Always according to the same Proposition, we have that the forms $\hat{\Xi}_g^{(g)}$ are cusp forms when we restrict to the Jacobian locus. Hence, when $g \leq 4$, their trace is 0 whenever we restrict to Jac_g. The extra contribution coming from $J^{(g)}$, with $c = c_0$, is added to get property (2) along $\mathcal{H}_g$. According to Corollary 19, $\Xi^{(5)}[0]$ verifies property (2) along Jac_5. Moreover, $\Xi^{(5)}[0]$ is the unique linear combination of theta series in $[\Gamma_5(1,2),8]$ that satisfies both properties (1) and (2). The cusp forms on Jac_5 from theta series are spanned by $\Xi^{(5)}[0]$ and $J^{(5)}$ and any $\Xi^{(5)}[0] + c\,J^{(5)}$ satisfies properties (1) but, since $J^{(5)}$ does not vanish identically on Jac_5, see [15], only $\Xi^{(5)}[0]$ also satisfies (2). □

7. Appendix: Theta series with harmonic polynomial coefficients

A different expression for the $\Xi^{(g)}[0]$, when $g \leq 4$, was obtained from theta series with harmonic polynomial coefficients. We briefly recall the results: Let X be a matrix with m rows and g columns. A harmonic form of weight ν in the matrix variable X is a polynomial $P(X)$ with the properties

$$\forall A \in \mathrm{GL}(n,\mathbb{C}), \quad P(XA) = (\det A)^{\nu} P(X),$$

$$\Delta P = \sum_{i,j} \frac{\partial^2}{(\partial X_{ij})^2} P = 0.$$

It can be proved, cf. [13], page 51 or [1], that if S is a positive definite integral unimodular quadratic form of degree m with 8 dividing m, then the theta series

$$\vartheta^{(g)}_{S,P}(\tau) = \sum_{X \in \mathbb{Z}^{m,g}} P(S^{1/2}X)\ e(1/2 \cdot \mathrm{tr}(S[X]\tau))$$

is a modular form of weight $m/2 + \nu$ relative to $\Gamma_g(1,2)$. It is a cusp form if $\nu > 0$. Moreover, if S is also even then we get a modular form relative to Γ_g.

A simple way to construct harmonic polynomials is the following: let L be a $m \times g$ matrix with $L'L = 0$ and $L'\overline{L} > 0$, then, for $\nu \in \mathbb{Z}_{\geq 0}$,

$$P_\nu(X) = \det(L'X)^{\nu}$$

is a harmonic polynomial of degree ν. Here $\overline{L}$ is the conjugate of L and necessarily $m \geq 2g$.

For $m = 8$, $\nu = 4$, $k = 8$, $S = I_g$ or E_8 and for $g = 1,2,3,4$ we choose L of the form $L' = \begin{pmatrix} 1_g & 0 & i1_g & 0 \end{pmatrix}$.

Proposition 21. *Let $m = 8$ and $\nu = 4$, then*

1. The theta series $\vartheta^{(g)}_{E_8,\,P_4}$ *vanish when* $g = 1, 2, 3$ *and, up to a nonzero multiplicative constant, is equal to* J *when* $g = 4$.

2. The theta series $\vartheta^{(g)}_{1_8,\,P_4}$ *do not vanish when* $g = 1, 2, 3, 4$.

Proof. In [23] and [26] the non-vanishing of $\vartheta^{(4)}_{E_8,\,P_4}$ has been proved. The vanishing of the other cases is a consequence of the general fact that there are no level one cusp forms of weight 8 when $g \le 3$. About the second statement, we observe that the nonvanishing is the consequence of a simple computation, since in all these cases, we have $a(\frac{1}{2}1_g) \neq 0$ for the Fourier coefficients of $\frac{1}{2}1_g$. In fact

$$a(\tfrac{1}{2}1_g) = \sum_{X \in \mathbb{Z}^{m,\,g}:\ X'X = 1_g} \det(L'X)^4.$$

For such X it is easy to check that $\det(L'X)$ is 0 or a fourth root of unity. Since there exist X such that the previous determinant is not zero, we get $a(\frac{1}{2}1_g) \neq 0$. □

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