Centralizer algebras of the group associated to $\mathbb{Z}_4$-codes

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Abstract

The purpose of this paper is to investigate the finite group which appears in the study of the Type II $\mathbf{Z}_4$-codes. To be precise, it is characterized in terms of generators and relations, and we determine the structure of the centralizer algebras of the tensor representations of this group.

1 Introduction

This is a continuation of the paper [7] in which the centralizer algebras of the tensor representations of the specified finite group are investigated. This group is closely connected to the Type II binary codes. The group to be investigated in this paper appears in the study of the Type II $\mathbb{Z}_4$-codes.

A problem dealt with in this paper is simple and quite natural. The theory of codes over $\mathbb{Z}_4$ has attracted great interest since around 1990 (*cf.* [6], [2], [1] and the references cited there). Let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ be the residue ring of the rational integers modulo 4. We take a $\mathbb{Z}_4$-code C of length n, that is, an additive subgroup of $\mathbb{Z}_4^n$. It is said to be Type II if

$$C = \{u \in \mathbb{Z}_4^n \ : \ (u, v) = \sum_i u_i v_i \equiv 0 \pmod 4, \ \forall v \in C\}$$

and

$$(v, v) \equiv 0 \pmod 8, \quad \forall v \in C.$$

The complete weight enumerator of C is

$$W_C(x, y, z, w) = \sum_{v \in C} x^{wt_0(v)} y^{wt_1(v)} z^{wt_2(v)} w^{wt_3(v)}$$

where $wt_a(v) = \sharp\{i : \ v_i = a\}$ for $v = (v_1, v_2, \ldots, v_n)$. In view of the relation with the theory of modular forms, the following type of invariants is more appropriate

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for our purpose (*cf.* [10], [1], [3], [8]). The symmetrized weight enumerator of C is defined as

$$S_C(x,y,z) = W_C(x,y,z,y).$$

Next we shall describe the relation between $S_C(x,y,z)$ and the invariant theory of the finite group.

Let $\eta = \dfrac{1+i}{\sqrt{2}}$ be a primitive 8-th root of unity and $\mathfrak{G}$ a group generated by $\mathcal{D} = \mathrm{diag}(1,\eta,-1)$ and

$$\mathcal{T} = \frac{\eta}{2}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}.$$

The group $\mathfrak{G}$ of order 384 natually acts on the polynomial ring $\mathbb{C}[x,y,z]$ of three variables over the complex number field $\mathbb{C}$: for $f \in \mathbb{C}[x,y,z]$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathfrak{G}$$

we have

$$Af(x,y,z) = f(a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z, a_{31}x + a_{32}y + a_{33}z).$$

Under this action we consider

$$\mathbb{C}[x,y,z]^{\mathfrak{G}} = \{f \in \mathbb{C}[x,y,z] : \ Af = f \text{ for any } A \in \mathfrak{G}\},$$

the invariant ring of $\mathfrak{G}$. It is easy to see that for a Type II $\mathbb{Z}_4$-code C the group $\mathfrak{G}$ keeps the symmetrized weight enumerator $S_C(x,y,z)$ stable. A remarkable fact is that the symmetrized weight enumerators are enough for the invariants of degrees multiple of 8. To be precise, let $\widetilde{\mathfrak{G}}$ be the group generated by $\mathcal{T}$, $\mathcal{D}$ and $\mathrm{diag}(\eta,\eta,\eta)$. Then $\widetilde{\mathfrak{G}}$ is of order 768 and the invariant ring $\mathbb{C}[x,y,z]^{\widetilde{\mathfrak{G}}}$ can be generated by the symmetrized weight enumerators of Type II $\mathbb{Z}_4$-codes ([2]). This implies that if we want to know the structure of the ring of some weight enumerators, then we have to know the invariant polynomial ring of some group in detail. This kind of fact, originally begun with Gleason [4], is one of the most fascinating theorems in coding theory, see [9].

On the other hand, as for the invariant theory we have a famous book [11] by H. Weyl which intensively studied the invariant theory as well as the representation theory. The commutator algebra plays an important role in the arguments of this book. We apply this philosophy to the group $\mathfrak{G}$. Let us consider the following situation: $\mathfrak{G}$ acts on covariant vectors $y^{(1)},\ldots,y^{(k)}$ cogrediently and on contravariant vectors $\xi^{(1)},\ldots,\xi^{(k)}$ contragradiently. Then the matrices $B = (b(i_1\cdots i_k; j_1\cdots j_k))$ of the coefficients of the invariant forms

$$\sum_{i;j} b(i_1\cdots i_k; j_1\cdots j_k)\xi^{(1)}_{i_1}\cdots\xi^{(k)}_{i_k} y^{(1)}_{j_1}\cdots y^{(k)}_{j_k}$$

of $\mathfrak{G}$ form the commutator algebra of the k-th tensor representation of $\mathfrak{G}$. Among other results, we explicitely determine this algebra.

We organize this paper as follows. First in Section 2 we give a presentation of $\mathfrak{G}$ by generators and relations. To determine the centralizer algebras, it is enough to consider the projective group $P\mathfrak{G} = \mathfrak{G}/Z(\mathfrak{G})$, the coset group by the center. We give the presentaion of $P\mathfrak{G}$ in Section 3. In Section 4, we afford a complete set of irreducible representations of $P\mathfrak{G}$. By Schur-Weyl duality, the multiplicities of the irreducible representations in the tensor space, determine the structure of the centralizer algebra. So we decompose the tensor representation of $P\mathfrak{G}$ into irreducible ones in Section 5. Finally in Section 6, we give the structure of the centralizer algebras of the tensor representations of $P\mathfrak{G}$.

2 Characterization

Let $\mathfrak{G}$ be the group introduced in Section 1. We note that $\mathcal{T}^2 = \mathrm{diag}(i, i, i)$ (and hence $\mathcal{T}^2$, $\mathcal{T}^4$ and $\mathcal{T}^6$) is in the center of $\mathfrak{G}$. We also note that both $\mathcal{D}$ and $\mathcal{T}$ have order 8 and that they satisfy the following relations:

$$\mathcal{T}\mathcal{D}\mathcal{T} = \mathcal{D}^7\mathcal{T}^3\mathcal{D}^7,\ \mathcal{T}\mathcal{D}^5\mathcal{T} = \mathcal{D}^3\mathcal{T}^7\mathcal{D}^3.$$

We show that actually these relations determine the group $\mathfrak{G}$, that is,

Theorem 2.1. *Let G be the group generated by the symbols D and T which obey the following relations:*

$$D^8 = 1, \tag{R1}$$
$$T^8 = 1, \tag{R2}$$
$$T^2D = DT^2, \tag{R3}$$
$$TDT = D^7T^3D^7, \tag{R4}$$
$$TD^5T = D^3T^7D^3. \tag{R5}$$

Then each element of G can be written in exactly one of the following forms:

$$1 \tag{W1}$$
$$D^{n_1}, \tag{W2}$$
$$T^{n_2}, \tag{W3}$$
$$D^{n_3}T^{n_4}, \tag{W4}$$
$$T^{p_1}D^{n_5}, \tag{W5}$$
$$D^{n_6}T^{p_2}D^{n_7}, \tag{W6}$$
$$TD^{e_1}T^{p_3}, \tag{W7}$$
$$D^{n_8}TD^{e_2}T^{p_2}. \tag{W8}$$

Here $n_1, \ldots, n_8 \in \{1, 2, \ldots, 7\}$, $p_1, p_2, p_3 \in \{1, 3, 5, 7\}$ and $e_1, e_2 \in \{2, 4, 6\}$. In particular the order of G is 384.

Before proving the theorem above, we shall show

Lemma 2.2. *We have*

$$TD^3T = D^5T^5D^5, \tag{R6}$$
$$TD^7T = DTD, \tag{R7}$$
$$D^{2i}TD^{2j}T = TD^{2j}TD^{2i}, \quad 2i, 2j \in \{2, 4, 6\}. \tag{R8}$$

Proof. From the relation (R5) we have

$$TD^5T = D^3T^7D^3 \Leftrightarrow TD^5 = D^3T^7D^3T^7 \Leftrightarrow D^5TD^5 = T^7D^3T^7 = TD^3T^5.$$

Multiplying T^4 from the right, we have $D^5T^5D^5 = TD^3T$.

From the relation (R4) we have

$$TDT = D^7T^3D^7 \Leftrightarrow TD = D^7T^3D^7T^7 = D^7TD^7T \Leftrightarrow DTD = TD^7T.$$

Finally we show (R8). For the case $2i = 2$, we have

$$\begin{aligned}
D^2TD^2T &= D \cdot DTD \cdot DT = D \cdot TD^7T \cdot DT = DTD \cdot D^6 \cdot TDT \\
&= TD^7T \cdot D^6 \cdot D^7T^3D^7 \\
&= TD^7T \cdot D^{13} \cdot T^3D^7 \\
&= TD^7 \cdot TD^5T \cdot T^2D^7 \\
&= TD^7 \cdot D^3T^7D^3 \cdot T^2D^7 \\
&= TD^{10}T^7T^2D^3D^7 \\
&= TD^2TD^2, \\
D^2TD^4T &= D \cdot DTD \cdot D^3T = D \cdot TD^7T \cdot D^3T = DTD \cdot D^6 \cdot TD^3T \\
&= TD^7T \cdot D^6 \cdot D^5T^5D^5 \\
&= TD^7T \cdot D^{11} \cdot T^5D^5 \\
&= TD^7 \cdot TD^3T \cdot T^4D^5 \\
&= TD^7 \cdot D^5T^5D^5 \cdot T^4D^5 \\
&= TD^{12}T^5T^4D^5D^5 \\
&= TD^4TD^2
\end{aligned}$$

and

$$\begin{aligned}
D^2TD^6T &= D \cdot DTD \cdot D^5T = D \cdot TD^7T \cdot D^5T = DTD \cdot D^6 \cdot TD^5T \\
&= TD^7T \cdot D^6 \cdot D^3T^7D^3 \\
&= TD^7T \cdot D^9 \cdot T^7D^3 \\
&= TD^7 \cdot TDT \cdot T^6D^3 \\
&= TD^7 \cdot D^7T^3D^7 \cdot T^6D^3
\end{aligned}$$

$$
\begin{aligned}
&= TD^{14}T^3T^6D^7D^3 \\
&= TD^6TD^2.
\end{aligned}
$$

For $2i = 4$, we utilize the case $2i = 2$ as follows.

$$
\begin{aligned}
D^4TD^2T &= D^2 \cdot D^2TD^2T = D^2 \cdot TD^2TD^2 = D^2TD^2T \cdot D^2 = TD^2TD^2 \cdot D^2 \\
&= TD^2TD^4.
\end{aligned}
$$

Similarly for $2i = 6$, we utilize the case $2i = 4$ and we can obtain the remaining relations. $\square$

Remark 2.3. 1) Because of (R3), we find that T^n is in the center of G if n is even.

2) The initial relations (R4), (R5) and the relations (R6), (R7) above show that

$$TD^{\mathrm{odd}}T = D^{\mathrm{odd}}T^{\mathrm{odd}}D^{\mathrm{odd}}.$$

Also we have

$$TD^{\mathrm{odd}} = D^{\mathrm{odd}}T^{\mathrm{odd}}D^{\mathrm{odd}}T^{\mathrm{odd}}.$$

Proof of Theorem 2.1. Suppose that $w \in G$ is written in one of the forms (W1)-(W8). We show that all the left and the right multiplications by the letters D and T are again written in one of the forms (W1)-(W8).

Case 1. The word in the form (W1) is the identity. It is obvious that the left and the right multiplications by D and T are written in the form (W2) and (W3) respectively.

Case 2. Let $w = D^{n_1}$. Then $Dw = wD = D^{n_1+1}$ is in the form (W2) or (W1). The left T-action $Tw = TD^{n_1}$ is in the form (W5) and the right T-action $wT = D^{n_1}T$ is in the form (W4).

Case 3. Let $w = T^{n_2}$. Then the left D-action $Dw = DT^{n_2}$ is in the form (W4). The right D-action $wD = T^{n_2}D$ is in the form (W5) if n_2 is odd, and $T^{n_2}D = DT^{n_2}$ is in the form (W4) if n_2 is even. The left and the right T-actions $Tw = wT = T^{n_2+1}$ is in the form (W3) or (W1).

Case 4. Let $w = D^{n_3}T^{n_4}$. Then the left D-action $Dw = D^{n_3+1}T^{n_4}$ is in the form (W4). The right D-action $wD = D^{n_3}T^{n_4}D$ is in the form (W6) if n_4 is odd. If n_4 is even, then $D^{n_3}T^{n_4}D = D^{n_3}DT^{n_4}$. This is in the form (W4). Next consider the left T-action. If n_4 is even, then $Tw = TD^{n_3}T^{n_4} = TT^{n_4}D^{n_3}$. This is in the form (W5). So we assume that n_4 is odd. If n_3 is even, then $TD^{n_3}T^{n_4}$ is in the form (W7). If n_3 is odd, then

$$TD^{n_3}T^{n_4} = TD^{n_3}T{\cdot}T^{\mathrm{even}} = D^{\mathrm{odd}}T^{\mathrm{odd}}D^{\mathrm{odd}}T^{\mathrm{even}} = D^{\mathrm{odd}}T^{\mathrm{odd}}T^{\mathrm{even}}D^{\mathrm{odd}}.$$

This is in the form (W6). If n_3 is even and n_4 is odd, then $TD^{n_3}T^{n_4}$ is in the form (W7). The right T-action $wT = D^{n_3}T^{n_4}T$ is in the form (W4).

Case 5. Let $w = T^{p_1}D^{n_5}$. The left D-action $Dw = DT^{p_1}D^{n_5}$ is in the form (W6) and the right D-action $wD = T^{p_1}D^{n_5}D$ is in the form (W5). The left T-action $Tw = TT^{p_1}D^{n_5} = T^{p_1+1}D^{n_5} = D^{n_5}T^{p_1+1}$. This is in the form (W4).

Consider the right T-action $wT = T^{p_1}D^{n_5}T$. If n_5 is even, this is in the form (W7). If n_5 is odd, then

$$T^{p_1}D^{n_5}T = T^{p_1-1}\cdot TD^{n_5}T = T^{p_1-1}\cdot D^{\mathrm{odd}}T^{\mathrm{odd}}D^{\mathrm{odd}} = D^{\mathrm{odd}}T^{p_1-1}T^{\mathrm{odd}}D^{\mathrm{odd}}.$$

This is in the form (W3) or (W6).

Case 6. Let $w = D^{n_6}T^{p_2}D^{n_7}$. Obviously, both the left and the right D-actions are in the form (W6). Consider the the left T-action $Tw = TD^{n_6}T^{p_2}D^{n_7}$. If both D^{n_6} and D^{n_7} are even, then by (R8),

$$\begin{aligned}TD^{n_6}T^{p_2}D^{n_7} &= TD^{n_6}TT^{p_2-1}D^{n_7} = TD^{n_6}TD^{n_7}\cdot T^{p_2-1} = D^{n_6}TD^{n_7}T\cdot T^{p_2-1}\\ &= D^{n_6}TD^{n_7}T^{p_2}.\end{aligned}$$

This is in the form (W8). If n_6 is odd, then

$$\begin{aligned}TD^{n_6}T^{p_2}D^{n_7} &= TD^{n_6}T\cdot T^{p_2-1}D^{n_7} = D^{\mathrm{odd}}T^{\mathrm{odd}}D^{\mathrm{odd}}T\cdot T^{p_2-1}D^{n_7}\\ &= D^{\mathrm{odd}}T^{\mathrm{odd}}T^{p_2-1}D^{\mathrm{odd}}T\cdot D^{n_7}.\end{aligned}$$

This is in the form (W6). If n_6 is even and n_7 is odd, then we have

$$\begin{aligned}TD^{n_6}T^{p_2}D^{n_7} &= TD^{n_6}TT^{p_2-1}D^{n_7} \qquad (1)\\ &= TD^{n_6}\cdot TD^{n_7}\cdot T^{p_2-1}\\ &= TD^{n_6}\cdot D^{\mathrm{odd}}T^{\mathrm{odd}}D^{\mathrm{odd}}T^{\mathrm{odd}}\cdot T^{p_2-1}\\ &= TD^{n_6}D^{\mathrm{odd}}T\cdot T^{\mathrm{odd}-1}D^{\mathrm{odd}}T^{\mathrm{odd}}T^{p_2-1}\\ &= D^{\mathrm{odd}}T^{\mathrm{odd}}D^{\mathrm{odd}}\cdot D^{\mathrm{odd}}T^{\mathrm{odd}-1}T^{\mathrm{odd}}T^{p_2-1}\\ &= D^{\mathrm{odd}}TT^{\mathrm{odd}-1}D^{\mathrm{odd}}D^{\mathrm{odd}}T^{\mathrm{odd}-1}T^{\mathrm{odd}}T^{p_2-1}\\ &= D^{\mathrm{odd}}TD^{\mathrm{odd}}D^{\mathrm{odd}}T^{\mathrm{odd}-1}T^{\mathrm{odd}-1}T^{\mathrm{odd}}T^{p_2-1}.\end{aligned}$$

This is in the form (W8). Consider the right T-action $wT = D^{n_6}T^{p_2}D^{n_7}T$. If n_7 is even, then we have $D^{n_6}T^{p_2}D^{n_7}T = D^{n_6}T\cdot T^{p_2-1}D^{n_7}T = D^{n_6}TD^{n_7}T^{p_2-1}T$. This is in the form $(W8)$. If n_7 is odd, then we have $D^{n_6}T^{p_2}D^{n_7}T = D^{n_6}T^{p_2-1}\cdot TD^{n_7}T = D^{n_6}T^{p_2-1}D^{\mathrm{odd}}T^{\mathrm{odd}}D^{\mathrm{odd}} = D^{n_6}D^{\mathrm{odd}}T^{p_2-1}T^{\mathrm{odd}}D^{\mathrm{odd}}$. This is in the form $(W6)$.

Case 7. Let $w = TD^{e_1}T^{p_3}$. The left D-action $Dw = DTD^{e_1}T^{p_3}$ is in the form (W8). Consider the right D-action $wD = TD^{e_1}T^{p_3}D$. We have already considered this case in the equation (1). The left T-action $Tw = TTD^{e_1}T^{p_3} = D^{e_1}T^{p_3+2}$ is in the form (W4). The right T-action $wT = TD^{e_1}T^{p_3}T = TD^{e_1}T^{p_3+1} = TT^{p_3+1}D^{e_1}$ is in the form (W5).

Case 8. Finally, consider the case $w = D^{n_8}TD^{e_2}T^{p_4}$. The left D-action $Dw = DD^{n_8}TD^{e_2}T^{p_4}$ is obviously in the form (W8). Consider the right D-action $wD = D^{n_8}TD^{e_2}T^{p_4}D$. As we have seen in the equation (1), $TD^{e_2}T^{p_4}D$ is in the form (W8). So is $D^{n_8}\cdot TD^{e_2}T^{p_4}D$. Consider the left T-action $Tw = TD^{n_8}TD^{e_2}T^{p_4}$. If n_8 is even, then we have

$$TD^{n_8}TD^{e_2}\cdot T^{p_4} = D^{e_2}TD^{n_8}T\cdot T^{p_4} = D^{e_2}TD^{n_8}T^{p_4+1} = D^{e_2}TT^{p_4+1}D^{n_8}.$$

This is in the form (W6). If n_8 is odd, then we have

$$\begin{aligned} TD^{n_8}T \cdot D^{e_2}T^{p_4} &= D^{\text{odd}}T^{\text{odd}}D^{\text{odd}} \cdot D^{e_2}T^{p_4} = D^{\text{odd}}TT^{\text{odd}-1}D^{\text{odd}}D^{e_2}T^{p_4} \\ &= D^{\text{odd}}TD^{\text{odd}}D^{e_2}T^{\text{odd}-1}T^{p_4}. \end{aligned}$$

This is in the form (W8). Finally consider the right T-action $wT = D^{n_8}TD^{e_2}T^{p_4}T = D^{n_8}TD^{e_2}T^{p_4+1} = D^{n_8}TT^{p_4+1}D^{e_2}$. This is in the form (W6).

Hence we found that under the relations (R1)-(R5), all possible words in the alphabet $\{D, T\}$ are written in the forms (W1)-(W8).

Consider the map ι from G to $\mathfrak{G}$ defined by $D \mapsto \mathcal{D}$ and $T \mapsto \mathcal{T}$. This is a well-defined group homomorphism, since $\mathfrak{G}$ respects the relations (R1)-(R5). This ι is surjective from the definition. It easy to check that all images of the words (W1)-(W8) are distinct. Hence ι is a group isomorphism. $\square$

By the derived isomorphism in the course of the proof above, we identify G with $\mathfrak{G}$.

3 Projective Group

Let Z be the center of G. Our objective is to analyze the structure of the centralizer algebra $\mathrm{End}_G(V^{\otimes k})$ of G in the tensor space. Since the center Z of G does not affect the structure of the centralizer, in the following, we have only to consider the projective group $PG = G/Z$ of G.

It is easy to see that $Z = \{1, T^2, T^4, T^6\}$. By the definition above, PG is generated by $\bar{D} = DZ$ and $\bar{T} = TZ$. By the similar argument in the previous section we find the defining relations and all words of PG.

Theorem 3.1. *$PG = G/Z$ has the following presentation:*
generators:

$$\bar{D}, \bar{T}$$

and relations

$$\bar{D}^8 = 1,\ \bar{T}^2 = 1,\ \bar{T}\bar{D}\bar{T} = \bar{D}^7\bar{T}\bar{D}^7,\ \bar{T}\bar{D}^5\bar{T} = \bar{D}^3\bar{T}\bar{D}^3. \tag{R0'}$$

Further, each element in PG can be written in exactly one of the following forms.

$$\bar{1}, \bar{D}^{n_1}, \bar{T}, \bar{D}^{n_1}\bar{T}, \bar{T}\bar{D}^{n_2}, \bar{D}^{n_3}\bar{T}\bar{D}^{n_4}, \bar{T}\bar{D}^{e_1}\bar{T}, \bar{D}^{n_5}\bar{T}\bar{D}^{e_2}\bar{T}. \tag{2}$$

Here $n_1, \ldots, n_5 \in \{1, 2, \ldots, 7\}$, and $e_1, e_2 \in \{2, 4, 6\}$.

Further, we can find that PG is divided into 10 conjugacy classes $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_{10}$, each of which is represented by

$$\bar{1}, \bar{D}, \bar{D}^2, \bar{D}^3, \bar{D}^4, \bar{D}^6, \bar{T}, \bar{D}\bar{T}, \bar{D}^4\bar{T}, \bar{D}^2\bar{T}\bar{D}^4\bar{T}.$$

4 The irreducible representations and the character table

In this section, we will find all the irreducible representations of PG and its character table. First we note that if we put $\bar{D} = \eta D$ and $\bar{T} = \eta^3 T$, then $\bar{D}$ and $\bar{T}$ satisfy the relations (R0'). In other words, the following map ρ_7 affords a representation of PG:

$$\rho_7(\bar{D}) = \eta D = \mathrm{diag}(\eta, i, -\eta), \quad \rho_7(\bar{T}) = \eta^3 T = \frac{-1}{2}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}.$$

We find the complex conjugate $\rho_6(\bar{D}) = \bar{\rho_7}(\bar{D}) = \mathrm{diag}(\eta^7, -i, \eta^3)$ and $\rho_6(\bar{T}) = \bar{\rho_7}(\bar{T}) = \rho_7(\bar{T})$ also afford an irreducible representation of PG. Moreover, the relations (R0') preserve the parity of the number of generators, we find that $\rho_2(\bar{D}) = \rho_2(\bar{T}) = -1$ afford a 1-dimensional representation other than a trivial representation ρ_1. Hence we have further two 3-dimensional irreducible representations, $\rho_8 = \rho_2 \otimes \rho_7$ and $\rho_9 = \rho_2 \otimes \rho_6$.

Next we consider $\rho_7 \otimes \rho_7$ and $\rho_7 \otimes \rho_6$. Let $\langle \boldsymbol{e}_i \otimes \boldsymbol{e}'_j \mid i, j = 1, 2, 3 \rangle$ be a basis of $V_7 \otimes V_7$ (resp. $V_7 \otimes V_6$). Then we have the following representation matrices of $\bar{D}$ and $\bar{T}$ respectively:

$$\begin{aligned}
\rho_7 \otimes \rho_7(\bar{D}) &= \mathrm{diag}(i, \eta^3, -i, \eta^3, -1, \eta^7, -i, \eta^7, i) \\
(\mathrm{resp.}\rho_7 \otimes \rho_6(\bar{D}) &= \mathrm{diag}(1, \eta^7, -1, \eta, 1, \eta^5, -1, \eta^3, 1)), \\
\rho_7 \otimes \rho_7(\bar{T}) &= \rho_7 \otimes \rho_6(\bar{T}) \\
&= \frac{1}{4}\begin{pmatrix}
1 & 2 & 1 & 2 & 4 & 2 & 1 & 2 & 1 \\
1 & 0 & -1 & 2 & 0 & -2 & 1 & 0 & -1 \\
1 & -2 & 1 & 2 & -4 & 2 & 1 & -2 & 1 \\
1 & 2 & 1 & 0 & 0 & 0 & -1 & -2 & -1 \\
1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\
1 & -2 & 1 & 0 & 0 & 0 & -1 & 2 & -1 \\
1 & 2 & 1 & -2 & -4 & -2 & 1 & 2 & 1 \\
1 & 0 & -1 & -2 & 0 & 2 & 1 & 0 & -1 \\
1 & -2 & 1 & -2 & 4 & -2 & 1 & -2 & 1
\end{pmatrix}.
\end{aligned}$$

If we put $\boldsymbol{v}_1 = \boldsymbol{e}_1 \otimes \boldsymbol{e}'_1 + \boldsymbol{e}_3 \otimes \boldsymbol{e}'_3$, $\boldsymbol{v}_2 = \boldsymbol{e}_1 \otimes \boldsymbol{e}'_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}'_1$ and $\boldsymbol{v}_3 = \boldsymbol{e}_2 \otimes \boldsymbol{e}'_2$, then we find $V_4 = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$ is $\rho_7^{\otimes 2}(\bar{T})$-invariant as well as $\rho_7^{\otimes 2}(\bar{D})$-invariant. Indeed, if we put $\rho_4 = \rho_7^{\otimes 2}|_{V_4}$, then we have

$$\rho_4(\bar{T})(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)\frac{1}{2}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix}$$

and

$$\rho_4(\bar{D})(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)\mathrm{diag}(i, -i, -1).$$

Note that no eigenvector of $\rho_4(\bar{D})$ is $\rho_4(\bar{T})$-invariant. This implies that (ρ_4, V_4) defines an irreducible representation of PG. Hence $\rho_5 = \rho_4 \otimes \rho_2$ also defines an irreducible representation.

Next we put $V_3 = \langle \boldsymbol{v}_4, \boldsymbol{v}_5 \rangle$, where $\boldsymbol{v}_4 = 2\boldsymbol{e}_1 \otimes \boldsymbol{e}_3' + 2\boldsymbol{e}_3 \otimes \boldsymbol{e}_1'$ and

$$\boldsymbol{v}_5 = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1' + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3' - \boldsymbol{e}_1 \otimes \boldsymbol{e}_3' - \boldsymbol{e}_3 \otimes \boldsymbol{e}_1' - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2'.$$

Then $\rho_3 = \rho_7 \otimes \rho_6|_{V_3}$ defines an irreducible representation of degree 2. The representation matrices with respect to this basis are

$$\rho_3(\bar{D})(\boldsymbol{v}_4, \boldsymbol{v}_5) = (\boldsymbol{v}_4, \boldsymbol{v}_5)\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho_3(\bar{T})(\boldsymbol{v}_4, \boldsymbol{v}_5) = (\boldsymbol{v}_4, \boldsymbol{v}_5)\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

It is easy to check that this representation is irreducible.

So far, we have got 9 irreducible representations of PG. The square sum of the dimensions of these irreducible representations is

$$1^2 + 1^2 + 2^2 + 3^2 + 3^2 + 3^2 + 3^2 + 3^2 + 3^2 = 60.$$

Since PG is of order 96 and has 10 conjugacy classes, there must be one more irreducible representation (ρ_{10}, V_{10}) of degree 6. Although we have not yet obtained the final representation, if we use the orthogonality of the characters we can find χ_{10}, the character of ρ_{10}, and obtain the character table of PG as follows.

PG	$\mathfrak{C}_1$ $\bar{1}$	$\mathfrak{C}_2$ $\bar{D}$	$\mathfrak{C}_3$ $\bar{D}^2$	$\mathfrak{C}_4$ $\bar{D}^3$	$\mathfrak{C}_5$ $\bar{D}^4$	$\mathfrak{C}_6$ $\bar{D}^6$	$\mathfrak{C}_7$ $\bar{T}$	$\mathfrak{C}_8$ $\bar{D}\bar{T}$	$\mathfrak{C}_9$ $\bar{D}^4\bar{T}$	$\mathfrak{C}_{10}$ $\bar{D}^2\bar{T}\bar{D}^4\bar{T}$
order	1	8	4	8	2	4	2	3	4	4
size	1	12	3	12	3	3	12	32	12	6
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1	1	-1	1
χ_3	2	0	2	0	2	2	0	-1	0	2
χ_4	3	-1	-1	-1	3	-1	1	0	1	-1
χ_5	3	1	-1	1	3	-1	-1	0	-1	-1
χ_6	3	$-i$	a	i	-1	b	-1	0	1	1
χ_7	3	i	b	$-i$	-1	a	-1	0	1	1
χ_8	3	$-i$	b	i	-1	a	1	0	-1	1
χ_9	3	i	a	$-i$	-1	b	1	0	-1	1
χ_{10}	6	0	2	0	-2	2	0	0	0	-2

Here $a = -1 - 2i$ and $b = -1 + 2i$.

Now we go back to the representation $\rho_7 \otimes \rho_6$. If we put

$$\begin{aligned} V_{10} = \langle &\boldsymbol{w}_3 = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1' - \boldsymbol{e}_3 \otimes \boldsymbol{e}_3', \boldsymbol{w}_4 = \boldsymbol{e}_1 \otimes \boldsymbol{e}_3' - \boldsymbol{e}_3 \otimes \boldsymbol{e}_1', \\ &\boldsymbol{w}_5 = \boldsymbol{e}_1 \otimes \boldsymbol{e}_2', \boldsymbol{w}_6 = \boldsymbol{e}_2 \otimes \boldsymbol{e}_1', \boldsymbol{w}_7 = \boldsymbol{e}_2 \otimes \boldsymbol{e}_3', \boldsymbol{w}_8 = \boldsymbol{e}_3 \otimes \boldsymbol{e}_2' \rangle, \end{aligned}$$

then V_{10} is both $\bar{T}$-invariant and $\bar{D}$-invariant. Hence V_{10} defines a representation ρ and we have

$$\rho(\bar{T})(\boldsymbol{w}_3,\dots,\boldsymbol{w}_8) = (\boldsymbol{w}_3,\dots,\boldsymbol{w}_8)\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 & 0 \end{pmatrix}$$

and

$$\rho(\bar{D})(\boldsymbol{w}_3,\dots,\boldsymbol{w}_8) = \mathrm{diag}(1,-1,\eta^7,\eta,\eta^5,\eta^3).$$

Since every character value of ρ at each conjugacy class coincides with that of χ_{10} of the character table of PG, $\rho=\rho_{10}$ gives the irreducible representation and we have thus obtained a complete set of representatives of the irreducible representations.

5 Decomposition of tensor representations

In the previous section, we have found the complete set of representatives of all irreducible representations of PG. In this section, we examine how the tensor powers of ρ_7 are decomposed into irreducible ones.

We follow the argument presented in the paper [7]. Let $\chi_1,\dots,\chi_{10}$ be the complete set of all irreducible characters of the group PG. Now suppose that we have a character

$$\chi = m_1\chi_1 + m_2\chi_2 + \cdots + m_{10}\chi_{10},$$

which has a value k_i at each conjugacy class $\mathfrak{C}_i,\ i=1,2,\dots,10$. Let $\boldsymbol{X}$ denote the matrix of the character table of PG, then we have

$$(m_1,\dots,m_{10}) = (k_1,\dots,k_{10})\boldsymbol{X}^{-1}. \tag{3}$$

In order to examine the structure of the centralizer algebra of $\rho_7^{\otimes k}$, the tensor power of the natural representation ρ_7, we decompose it into the irreducible ones. For this, we need to decompose $\rho_7\otimes\rho_i$ $(i=1,2,\dots,10)$ one by one.

By the argument and/or the character table in the previous section, we already have the following:

$$\chi_7\cdot\chi_1 = \chi_7,$$
$$\chi_7\cdot\chi_2 = \chi_8.$$

Using the equation (3), further we have

$$\chi_7\cdot\chi_3 = \chi_7+\chi_8,$$
$$\chi_7\cdot\chi_4 = \chi_6+\chi_{10},$$

$$
\begin{aligned}
\chi_7 \cdot \chi_5 &= \chi_9 + \chi_{10}, \\
\chi_7 \cdot \chi_6 &= \chi_1 + \chi_3 + \chi_{10}, \\
\chi_7 \cdot \chi_7 &= \chi_4 + \chi_6 + \chi_9, \\
\chi_7 \cdot \chi_8 &= \chi_5 + \chi_6 + \chi_9, \\
\chi_7 \cdot \chi_9 &= \chi_2 + \chi_3 + \chi_{10}, \\
\chi_7 \cdot \chi_{10} &= \chi_4 + \chi_5 + \chi_7 + \chi_8 + \chi_{10}.
\end{aligned}
$$

For example, we take up the $\chi_7 \cdot \chi_{10}$ case. It is easy to compute

$$\chi_7 \cdot \chi_{10}(\mathfrak{C}_1, \ldots, \mathfrak{C}_{10}) = (18, 0, -2+4i, 0, 2, -2-4i, 0, 0, 0, -2).$$

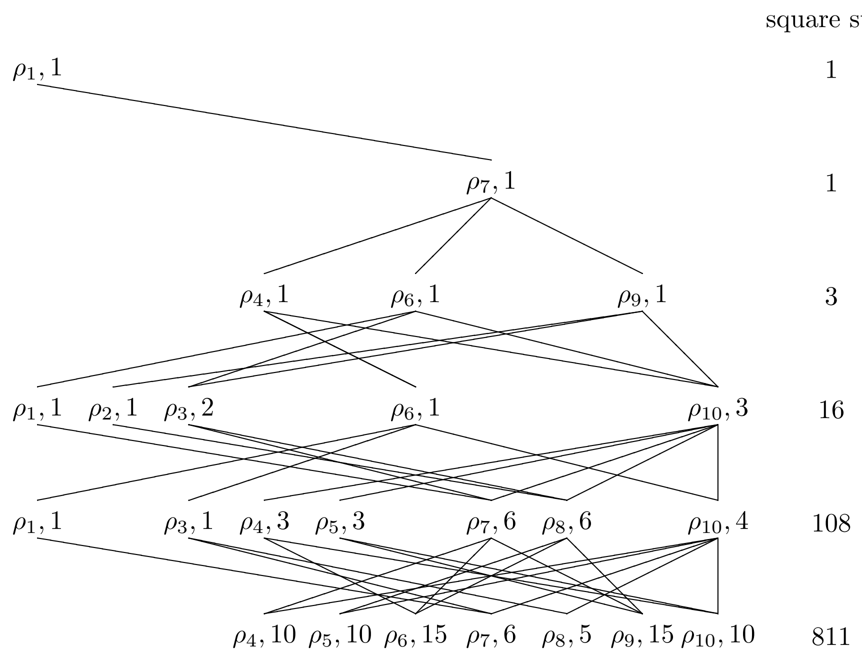
Then we compute

$$(18, 0, -2+4i, 0, 2, -2-4i, 0, 0, 0, -2)\boldsymbol{X}^{-1} = (0, 0, 0, 1, 1, 0, 1, 1, 0, 1)$$

and this means, due to the identity (3),

$$\chi_7 \cdot \chi_{10} = \chi_4 + \chi_5 + \chi_7 + \chi_8 + \chi_{10}.$$

By the above calculation, we obtain the Bratteli diagram of the decomposition of $\rho_7^{\otimes k}$ into irreducible ones. (For the Bratteli diagram, see for example Goodman-de la Harpe-Jones [5], §2.3.) Accordingly, the square sum of the multiplicities on the k-th row is the dimension of $\mathrm{End}_{PG}(V_7^{\otimes k})$.



6 Centralizer algebra

In the previous section, we have seen that the dimensions of $\mathcal{A}_k = \mathrm{End}_{PG}(V_7^{\otimes k})$ $(k = 0, 1, 2, \ldots, 5)$ are 1, 1, 3, 16, 108, 811. The number of paths from the top vertex to the bottom vertices on the Hasse diagram will be calculated using the adjacent matrix of the diagram. To be explicit, the formulae

$$\chi_7(\mathfrak{C}_j)\cdot\chi_i(\mathfrak{C}_j)=\sum_{k=1}^{10} m_{ik}\chi_k(\mathfrak{C}_j),\quad i,j=1,2,\ldots,10$$

obtained in the previous section lead to

$$\boldsymbol{X}\operatorname{diag}(\chi_7(\mathfrak{C}_1),\ldots,\chi_7(\mathfrak{C}_{10}))=A\boldsymbol{X}$$

or

$$\boldsymbol{X}\operatorname{diag}(\chi_7(\mathfrak{C}_1),\ldots,\chi_7(\mathfrak{C}_{10}))\boldsymbol{X}^{-1}=A \tag{4}$$

where

$$A=\begin{pmatrix}0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&1&0&0\\0&0&0&0&0&1&0&0&0&1\\0&0&0&0&0&0&0&0&1&1\\1&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&1&0&0&1&0\\0&0&0&0&1&1&0&0&1&0\\0&1&1&0&0&0&0&0&0&1\\0&0&0&1&1&0&1&1&0&1\end{pmatrix}.$$

Let $d_1(k),\ldots,d_{10}(k)$ be the number of paths at the bottom (the k-th floor) vertices of the Hasse diagram. Here k starts from 0 and the floor on which the first ρ_1 lies is the 0-th floor. Then the $d_\ell(k)$'s are equal to the multiplicities of the irreducible components of PG in $\operatorname{End}(V_7^{\otimes k})$:

$$\chi_7^k=d_1(k)\chi_1+d_2(k)\chi_2+\cdots+d_{10}(k)\chi_{10}.$$

It is easy to see

$$(d_1(k),\ldots,d_{10}(k))=(1,0,0,0,0,0,0,0,0,0)A^k\quad(k\geq 0).$$

By the relation (4), however, they are calculated by the character table $\boldsymbol{X}$ of PG as follows.

$$(d_1(k),\ldots,d_{10}(k))=(1,0,\ldots,0)\boldsymbol{X}\operatorname{diag}(\chi_7(\mathfrak{C}_1)^k,\ldots,\chi_7(\mathfrak{C}_{10})^k)\boldsymbol{X}^{-1}.$$

Hence we have for $k\geq 1$

$$\begin{aligned}
d_1(k)&=\frac{3^k}{96}+\frac{a^k}{32}+\frac{b^k}{32}+\frac{5\cdot(-1)^k}{32}+\frac{3}{16}+\frac{i^k}{8}+\frac{(-i)^k}{8},\\
d_2(k)&=\frac{3^k}{96}+\frac{a^k}{32}+\frac{b^k}{32}-\frac{3\cdot(-1)^k}{32}-\frac{1}{16}-\frac{i^k}{8}-\frac{(-i)^k}{8},\\
d_3(k)&=\frac{3^k}{48}+\frac{a^k}{16}+\frac{b^k}{16}+\frac{(-1)^k}{16}+\frac{1}{8},
\end{aligned}$$

$$d_4(k) = \frac{3^k}{32} - \frac{a^k}{32} - \frac{b^k}{32} + \frac{7\cdot(-1)^k}{32} + \frac{1}{16} - \frac{i^k}{8} - \frac{(-i)^k}{8},$$
$$d_5(k) = \frac{3^k}{32} - \frac{a^k}{32} - \frac{b^k}{32} - \frac{(-1)^k}{32} - \frac{3}{16} + \frac{i^k}{8} + \frac{(-i)^k}{8},$$
$$d_6(k) = \frac{3^k}{32} + \frac{a\cdot a^k}{32} + \frac{b\cdot b^k}{32} - \frac{5\cdot(-1)^k}{32} + \frac{3}{16} + \frac{i\cdot i^k}{8} - \frac{i\cdot(-i)^k}{8},$$
$$d_7(k) = \frac{3^k}{32} + \frac{b\cdot a^k}{32} + \frac{a\cdot b^k}{32} - \frac{5\cdot(-1)^k}{32} + \frac{3}{16} - \frac{i\cdot i^k}{8} + \frac{i\cdot(-i)^k}{8},$$
$$d_8(k) = \frac{3^k}{32} + \frac{b\cdot a^k}{32} + \frac{a\cdot b^k}{32} + \frac{3\cdot(-1)^k}{32} - \frac{1}{16} + \frac{i\cdot i^k}{8} - \frac{i\cdot(-i)^k}{8},$$
$$d_9(k) = \frac{3^k}{32} + \frac{a\cdot a^k}{32} + \frac{b\cdot b^k}{32} + \frac{3\cdot(-1)^k}{32} - \frac{1}{16} - \frac{i\cdot i^k}{8} + \frac{i\cdot(-i)^k}{8},$$
$$d_{10}(k) = \frac{3^k}{16} + \frac{a^k}{16} + \frac{b^k}{16} - \frac{(-1)^k}{16} - \frac{1}{8}.$$

Here $a = -1 - 2i$ and $b = -1 + 2i$. We shall summarize our results in the following way.

Theorem 6.1. *Let $\mathcal{A}_k = End_{PG}(V_7^{\otimes k})$ be a centralizer algebra of PG in $V_7^{\otimes k}$, where PG acts on $V_7^{\otimes k}$ diagonally. Then $\mathcal{A}_k$ has the following multi-matrix structure:*

$$\mathcal{A}_k \cong \begin{cases} \mathbb{C} & (k=0), \\ \bigoplus_{\ell=1}^{10} M_{d_\ell(k)}(\mathbb{C}) & (k \geq 1) \end{cases}$$

in which the $d_\ell(k)$'s are explicitly determined above.

Calculating the square sum of the dimensions of the simple components of $\mathcal{A}_k$ we finally derive the following

Corollary 6.2. *We have*

$$\dim \mathcal{A}_k = \begin{cases} 1 & (k=0), \\ \dfrac{57 + 6\cdot 5^k + 9^k}{96} & (k \geq 1). \end{cases}$$

We conclude this paper with a small table of the values $\dim \mathcal{A}_k$.

k	0	1	2	3	4	5	6	7	8	9
$\dim \mathcal{A}_k$	1	1	3	16	108	811	6513	54706	472818	4157701

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