

WEIGHT ENUMERATORS, INTERSECTION ENUMERATORS AND JACOBI POLYNOMIALS II

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ABSTRACT. In the present paper, we introduce the concepts of Jacobi polynomials and intersection enumerators of codes over \mathbb{F}_q and \mathbb{Z}_k for arbitrary genus g . We also discuss the interrelation among them. Finally, we give the MacWilliams type identities for Jacobi polynomials.

Key Words: Code, weight enumerator, intersection enumerator, Jacobi polynomial.

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1. INTRODUCTION

Weight enumerators make relationships between coding theory, invariant theory, and modular forms [3, 4, 5, 9, 11, 13, 15, 19, 20]. A striking generalization of the weight enumerators was obtained by Ozeki [18], who gave the concept of Jacobi polynomials for codes in analogy with Jacobi forms [10] of lattices and presented a generalization of the MacWilliams identity. In [16], the notion of the intersection polynomials was given for some computations of extremal codes.

In [17], the concepts of genus g Jacobi polynomials and intersection enumerators of binary codes were introduced and the MacWilliams type identities of Jacobi polynomials were given. Moreover, the concept of genus g (homogeneous) Jacobi polynomials of binary codes in [17] was generalized to the notion of g -fold joint (homogeneous) Jacobi polynomials of codes over \mathbb{F}_q and \mathbb{Z}_k in [6] and the MacWilliams type identity for the generalized notion was presented. In the present paper, we give the generalizations of the concepts discussed in [17], such as genus g Jacobi polynomials and genus g intersection polynomials of binary codes, to some non-binary cases, like for codes over \mathbb{F}_q and \mathbb{Z}_k . Further, we

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generalize some of the results in [17], particularly, the MacWilliams type identities for Jacobi polynomials.

Throughout this paper, we assume that \mathfrak{R} denotes either the finite field \mathbb{F}_q of order q , where q is a prime power or the ring \mathbb{Z}_k of integers modulo k for some positive integer $k \geq 2$. Moreover, \mathfrak{R}^* denotes the set of non-zero elements of \mathfrak{R} . We prefer to call the elements of \mathfrak{R}^n as *vectors*. For $u = (u_1, \dots, u_n) \in \mathfrak{R}^n$, we denote $\text{wt}_\ell(u) := \#\{i \mid u_i = \ell\}$.

The purpose of this paper is to introduce the following polynomials and discuss their properties. Let $[g] := \{1, \dots, g\}$ and we denote by $n_a(u_1, \dots, u_g)$ the number of i such that $a = (u_{1,i}, \dots, u_{g,i})$ for $u_1, \dots, u_g \in \mathfrak{R}^n$ and $a \in \mathfrak{R}^g$.

Definition 1.1. Let g be a positive integer and C be an \mathfrak{R} -linear code of length n . Let $\binom{[g]}{p} := \{(K_1, \dots, K_p) \in \mathbb{Z}^p \mid 1 \leq K_1 < \dots < K_p \leq g\}$ for any positive integer p such that $1 \leq p \leq g$.

- (1) The g -th *weight enumerator* of C is

$$W_C^{(g)}(\{x_a\}_{a \in \mathfrak{R}^g}) = \sum_{u_1, \dots, u_g \in C} \prod_{a \in \mathfrak{R}^g} x_a^{n_a(u_1, \dots, u_g)}.$$

- (2) The g -th *intersection enumerator* of C is

$$\begin{aligned} I_C^{(g)}(\{X_{K,L}\}_{1 \leq p \leq g, K \in \binom{[g]}{p}, L \in (\mathfrak{R}^*)^p}) \\ = \sum_{u_1, \dots, u_g \in C} \prod_{1 \leq p \leq g} \prod_{K \in \binom{[g]}{p}} \prod_{L \in (\mathfrak{R}^*)^p} X_{K,L}^{n_L(u_{K_1}, \dots, u_{K_p})}. \end{aligned}$$

- (3) The g -th *homogeneous Jacobi polynomial* of C with reference vector $v \in \mathfrak{R}^n$ is

$$\mathfrak{Jac}_{C,v}^{(g)}(\{y_a\}_{a \in \mathfrak{R}^{g+1}}) = \sum_{u_1, \dots, u_g \in C} \prod_{a \in \mathfrak{R}^{g+1}} y_a^{n_a(u_1, \dots, u_g, v)}.$$

- (4) Assume that $X_{(g+1),(j)} = 1$ for all $j \in \mathfrak{R}^*$. The g -th *inhomogeneous Jacobi polynomial* of C with reference vector $v \in \mathfrak{R}^n$ is

$$\begin{aligned} \text{Jac}_{C,v}^{(g)}(\{X_{K,L}\}_{1 \leq p \leq g+1, K \in \binom{[g+1]}{p}, L \in (\mathfrak{R}^*)^p}) \\ = \sum_{u_1, \dots, u_g \in C} \prod_{1 \leq p \leq g+1} \prod_{\substack{K \in \binom{[g+1]}{p} \\ u_{g+1} = v}} \prod_{L \in (\mathfrak{R}^*)^p} X_{K,L}^{n_L(u_{K_1}, \dots, u_{K_p})}. \end{aligned}$$

Equivalently, we can also write the g -th *inhomogeneous Jacobi polynomial* of C with reference vector $v \in \mathfrak{R}^n$ as follows:

$$\begin{aligned} & \text{Jac}_{C,v}^{(g)}(\{X_{(k),(\ell)}\}_{k \in [g], \ell \in \mathfrak{R}^*}, \{X_{K,L}\}_{2 \leq p \leq g+1, K \in \binom{[g+1]}{p}, L \in (\mathfrak{R}^*)^p}) \\ &= \sum_{u_1, \dots, u_g \in C} \left(\prod_{k \in [g]} \prod_{\ell \in \mathfrak{R}^*} X_{(k),(\ell)}^{n_\ell(u_k)} \right) \\ & \quad \left(\prod_{2 \leq p \leq g+1} \prod_{\substack{K \in \binom{[g+1]}{p} \\ u_{g+1} = v}} \prod_{L \in (\mathfrak{R}^*)^p} X_{K,L}^{n_L(u_{K_1}, \dots, u_{K_p})} \right). \end{aligned}$$

Note that when we say $u_1, \dots, u_g \in C$ in the definitions of the four polynomials, we mean that the vectors u_i 's are not necessary to be distinct.

Example 1.1. Let C_2 be an \mathbb{F}_3 -linear code with length 4 having the codewords: $(0, 0), (1, 1), (2, 2)$. The following examples will make the polynomials in Definition 1.1 that are in complicated looking for a simple concept, easy to understand. We derive the polynomials for $g = 2$. Let $v = (1, 2) \in \mathbb{F}_3^2$ be the reference vector for the homogeneous and inhomogeneous Jacobi polynomials.

$$\begin{aligned} W_{C_4}^{(2)}(\{x_a\}_{a \in \mathbb{F}_3^2}) &= x_{(0,0)}^2 + x_{(1,0)}^2 + x_{(2,0)}^2 + x_{(0,1)}^2 + x_{(1,1)}^2 + x_{(2,1)}^2 + x_{(0,2)}^2 \\ & \quad + x_{(1,2)}^2 + x_{(2,2)}^2 \end{aligned}$$

$$\begin{aligned} I_{C_4}^{(2)}(\{X_{K,L}\}_{1 \leq p \leq 2, K \in \binom{[2]}{p}, L \in (\mathbb{F}_3^*)^p}) &= 1 + X_{(1),(1)}^2 + X_{(1),(2)}^2 + X_{(2),(1)}^2 \\ & \quad + X_{(1),(1)}^2 X_{(2),(1)}^2 X_{(1,2),(1,1)}^2 + X_{(1),(2)}^2 X_{(2),(1)}^2 X_{(1,2),(2,1)}^2 \\ & \quad + X_{(1),(1)}^2 X_{(2),(2)}^2 X_{(1,2),(1,2)}^2 + X_{(1),(2)}^2 X_{(2),(2)}^2 X_{(1,2),(2,2)}^2 + X_{(2),(2)}^2 \end{aligned}$$

$$\begin{aligned} \mathfrak{J}ac_{C_4,v}^{(2)}(\{y_a\}_{a \in \mathbb{F}_3^2}) &= y_{(0,0,1)}^1 y_{(0,0,2)}^1 + y_{(1,0,1)}^1 y_{(1,0,2)}^1 + y_{(2,0,1)}^1 y_{(2,0,2)}^1 \\ & \quad + y_{(0,1,1)}^1 y_{(0,1,2)}^1 + y_{(1,1,1)}^1 y_{(1,1,2)}^1 + y_{(2,1,1)}^1 y_{(2,1,2)}^1 \\ & \quad + y_{(0,2,1)}^1 y_{(0,2,2)}^1 + y_{(1,2,1)}^1 y_{(1,2,2)}^1 + y_{(2,2,1)}^1 y_{(2,2,2)}^1 \end{aligned}$$

$$\begin{aligned}
Jac_{C,v}^{(2)}(\{X_{K,L}\}_{1 \leq p \leq 3, K \in \binom{[3]}{p}, L \in (\mathbb{F}_3^*)^p}) &= 1 + X_{(1),(1)}^2 X_{(1,3),(1,1)}^1 X_{(1,3),(1,2)}^1 \\
&+ X_{(1),(2)}^2 X_{(1,3),(2,1)}^1 X_{(1,3),(2,2)}^1 + X_{(2),(1)}^2 X_{(2,3),(1,1)}^1 X_{(2,3),(1,2)}^1 \\
&+ X_{(1),(1)}^2 X_{(2),(1)}^2 X_{(1,2),(1,1)}^2 X_{(1,3),(1,1)}^1 X_{(1,3),(1,2)}^1 X_{(2,3),(1,1)}^1 X_{(2,3),(1,2)}^1 \\
&\quad X_{(1,2,3),(1,1,1)}^1 X_{(1,2,3),(1,1,2)}^1 \\
&+ X_{(1),(2)}^2 X_{(2),(1)}^2 X_{(1,2),(2,1)}^2 X_{(1,3),(2,1)}^1 X_{(1,3),(2,2)}^1 X_{(2,3),(1,1)}^1 X_{(2,3),(1,2)}^1 \\
&\quad X_{(1,2,3),(2,1,1)}^1 X_{(1,2,3),(2,1,2)}^1 \\
&+ X_{(2),(2)}^2 X_{(2,3),(2,1)}^1 X_{(2,3),(2,2)}^1 + X_{(1),(1)}^2 X_{(2),(2)}^2 X_{(1,2),(1,2)}^2 X_{(1,3),(1,1)}^1 \\
&\quad X_{(1,3),(1,2)}^1 X_{(2,3),(2,1)}^1 X_{(2,3),(2,2)}^1 X_{(1,2,3),(1,2,1)}^1 X_{(1,2,3),(1,2,2)}^1 \\
&+ X_{(1),(2)}^2 X_{(2),(2)}^2 X_{(1,2),(2,2)}^2 X_{(1,3),(2,1)}^1 X_{(1,3),(2,2)}^1 X_{(2,3),(2,1)}^1 X_{(2,3),(2,2)}^1 \\
&\quad X_{(1,2,3),(2,2,1)}^1 X_{(1,2,3),(2,2,2)}^1
\end{aligned}$$

If there is no confusion, we prefer to write the notations of the above said polynomials in a simple form by omitting the notations of the variables in the polynomials as $W_C^{(g)}$, $I_C^{(g)}$, $\mathfrak{Jac}_{C,v}^{(g)}$ and $Jac_{C,v}^{(g)}$.

Remark 1.1. We have the following remarks:

(1) It is easy to see that

$$\begin{cases} \mathfrak{Jac}_{C,0}^{(g)} = W_C^{(g)} \\ Jac_{C,0}^{(g)} = I_C^{(g)}. \end{cases}$$

(2) The number of variables in each polynomial is given by

$$\begin{cases} W_C^{(g)} & : |\mathfrak{A}|^g, \\ I_C^{(g)} & : \sum_{p=1}^g \binom{g}{p} (|\mathfrak{A}| - 1)^p = |\mathfrak{A}|^g - 1, \\ \mathfrak{Jac}_{C,v}^{(g)} & : |\mathfrak{A}|^{g+1}, \\ Jac_{C,v}^{(g)} & : \sum_{p=1}^{g+1} \binom{g+1}{p} (|\mathfrak{A}| - 1)^p - (|\mathfrak{A}| - 1) \\ & = |\mathfrak{A}| (|\mathfrak{A}|^g - 1). \end{cases}$$

This paper is organized as follows. In Section 2, we give definitions and some basic properties of codes used in this paper. In Section 3, we give relations between the four polynomials (Theorem 3.1, Theorem 3.2, Theorem 3.3). In Section 4, we give the MacWilliams type identities for a g -th homogeneous Jacobi polynomial (Theorem 4.1) and a g -th inhomogeneous Jacobi polynomial (Theorem 4.2).

2. PRELIMINARIES

We refer the readers to [1, 12, 14] for the background of the coding theory. Let \mathbb{F}_q be the finite field of order q , where $q = p^f$ for some

prime number p . Then \mathbb{F}_q^n denotes the n -dimensional vector space over \mathbb{F}_q equipped with the following inner product

$$u \cdot v := \sum_{i=1}^n u_i v_i,$$

where $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in \mathbb{F}_q^n$. If q is an even power f of an arbitrary prime p , then it is convenient to consider another inner product given by

$$u \cdot v := \sum_{i=1}^n u_i v_i \sqrt{q}$$

On the other hand, let \mathbb{Z}_k be the finite ring of integers modulo k for some positive integer $k \geq 2$. Then \mathbb{Z}_k^n denotes the \mathbb{Z}_k -module of all n -tuples over \mathbb{Z}_k . Let $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$ be two elements of \mathbb{Z}_k^n . Then the inner product of u and v on \mathbb{Z}_k^n is defined as

$$u \cdot v := \sum_{i=1}^n u_i v_i.$$

An \mathfrak{R} -linear code C of length n is either a vector subspace of \mathfrak{R}^n when \mathfrak{R} represents \mathbb{F}_q or a submodule of \mathfrak{R}^n if \mathfrak{R} denotes \mathbb{Z}_k . The dual of an \mathfrak{R} -linear code C is denoted by C^\perp and defined as

$$C^\perp := \{v \in \mathfrak{R}^n \mid u \cdot v = 0 \text{ for all } u \in C\}.$$

Let $a = (a_1, \dots, a_g) \in \mathfrak{R}^g$ with nonzero weight p . Then $\text{Vsupp}(a) := (i \mid a_i \neq 0)$, where i 's are in ascending order, is an element of $\binom{[g]}{p}$. Let $K = (K_1, \dots, K_p) \in \binom{[g]}{p}$. Then by $k \in K$ we mean that $k = K_j$ for some $1 \leq j \leq p$. Moreover, by $K' \subset K$, we denote either a r -tuple $K' = (K_{m_1}, \dots, K_{m_r})$ for $1 \leq r \leq p$ such that $1 \leq m_1 \leq \dots \leq m_r \leq p$ or an empty tuple K' which we prefer to write as \emptyset .

Now we have the following useful lemma for this paper.

Lemma 2.1. *Let u_1, \dots, u_g be elements of \mathfrak{R}^n . Then the following hold.*

$$(1) \quad \prod_{\substack{a \in \mathfrak{R}^g, \\ a \neq 0}} \prod_{\substack{K \subset \text{Vsupp}(a), \\ K \neq \emptyset}} X_{K, a_K}^{n_a(u_1, \dots, u_g)} \\ = \prod_{1 \leq p \leq g} \prod_{K \in \binom{[g]}{p}} \prod_{L \in (\mathfrak{R}^*)^p} X_{K, L}^{n_L(u_{K_1}, \dots, u_{K_p})},$$

where a_K is the length $|K|$ vector indexed by K such that $a_K = (a_{i_1}, \dots, a_{i_{|K|}})$ with $i_1 < \dots < i_{|K|}$.

$$(2) \text{ For } 0 \neq a \in \mathfrak{R}^g, x_a = x_0 \prod_{\substack{K \subset \text{Vsupp}(a), \\ K \neq \emptyset}} \prod_{K' \subset K} x_{a_{K'}}^{(-1)^{|K|-|K'|}}, \text{ where}$$

$a_{K'}$ is the length g vector such that $a_{K',i} = a_i$ for $i \in K'$, $a_{K',i} = 0$ for $i \notin K'$, and $a_\emptyset = 0$.

Proof. (1) The left-hand side counts for $1 \leq i \leq n$, the number of pairs $\{(K, a_K) \mid K \subset \text{Vsupp}(u_{1,i}, \dots, u_{g,i})\}$. The right-hand side counts for $1 \leq i \leq g$, the number of pairs $\{(K, L) \mid K \subset \text{Vsupp}(u_{i,1}, \dots, u_{i,n})\}$.

(2) Let the exponent of $x_{a_{K'}}$ in the right-hand be $E(x_{a_{K'}})$. It is immediate that $x_{a_{K'}}$ appears only once for $K' = \text{Vsupp}(a)$. Let the length of $\text{Vsupp}(a)$ and K' be m and ℓ , respectively. Then for $K' \neq \text{Vsupp}(a)$ with $\ell \neq 0$, we have

$$\begin{aligned} E(x_{a_{K'}}) &= \sum_{k=\ell}^m \#\{K \in \binom{[g]}{k} \mid K' \subset K \subset \text{Vsupp}(a)\} \\ &= \sum_{k=\ell}^m (-1)^{k-\ell} \binom{m-\ell}{k-\ell} \\ &= \sum_{t=0}^{m-\ell} (-1)^t \binom{m-\ell}{t} \\ &= (1 + (-1))^{m-\ell} \\ &= 0. \end{aligned}$$

Now for $K' \neq \text{Vsupp}(a)$ with $\ell = 0$, we have

$$\begin{aligned} E(x_{a_{K'}}) &= 1 + \sum_{k=1}^m \#\{K \in \binom{[g]}{k} \mid K \subset \text{Vsupp}(a)\} \\ &= 1 + \sum_{k=1}^m (-1)^k \binom{m}{k} \\ &= (1 + (-1))^m \\ &= 0. \end{aligned}$$

Thus the above discussed sums conclude that the right-hand side consists of x_a only. This completes the proof. \square

3. RELATIONS BETWEEN FOUR POLYNOMIALS

3.1. Weight enumerators and Intersection enumerators. In this section, we give a relation between weight enumerators and intersection enumerators.

Theorem 3.1. *Let C be an \mathfrak{R} -linear code of length n . Then the following hold.*

$$(1) W_C^{(g)} \left(x_0 \leftarrow 1, x_a \leftarrow \prod_{\substack{K \subset \text{Vsupp}(a), \\ K \neq \emptyset}} X_{K, a_K} \text{ for } a \neq 0 \right) = I_C^{(g)},$$

where $a \in \mathfrak{R}^g$, and a_K is the length $|K|$ vector indexed by K such that $a_K = (a_{i_1}, \dots, a_{i_{|K|}})$ with $i_1 < \dots < i_{|K|}$.

$$(2) x_0^n I_C^{(g)} \left(X_{K, L} \leftarrow \prod_{K' \subset K} x_{L_{K'}}^{(-1)^{|K| - |K'|}} \right) = W_C^{(g)},$$

where $K = (K_1, \dots, K_p) \in \binom{[g]}{p}$ for $1 \leq p \leq g$, and $K' = (K_{m_1}, \dots, K_{m_r})$ for $1 \leq r \leq p$ and $1 \leq m_1 \leq \dots \leq m_r \leq p$, and $L_{K'}$ is the length g vector such that $L_{K', i} = L_j$ for $i = K_{m_j} \in K'$ where $1 \leq j \leq r$, $L_{K', i} = 0$ for $i \notin K'$, and $L_\emptyset = 0$.

Proof. (1) By Lemma 2.1 (1),

$$\begin{aligned} & W_C^{(g)} \left(x_0 \leftarrow 1, x_a \leftarrow \prod_{\substack{K \subset \text{Vsupp}(a) \\ K \neq \emptyset}} X_{K, a_K} \text{ for } a \neq 0 \right) \\ &= \sum_{u_1, \dots, u_g \in C} \left(\prod_{\substack{a \in \mathfrak{R}^g \\ a \neq 0}} \prod_{\substack{K \subset \text{Vsupp}(a), \\ K \neq \emptyset}} X_{K, a_K}^{n_a(u_1, \dots, u_g)} \right) \\ &= \sum_{u_1, \dots, u_g \in C} \prod_{1 \leq p \leq g} \prod_{K \in \binom{[g]}{p}} \prod_{L \in (\mathfrak{R}^*)^p} X_{K, L}^{n_L(u_{K_1}, \dots, u_{K_p})} \\ &= I_C^{(g)}. \end{aligned}$$

(2) Let $u_1, \dots, u_g \in \mathfrak{X}^n$ and $a \in \mathfrak{X}^g$. We observe for $a \neq 0$ that $x_0^n = x_0^{n_0(u_1, \dots, u_g)} \prod_{a \in \mathfrak{X}^g} x_0^{n_a(u_1, \dots, u_g)}$. Now by Lemma 2.1,

$$\begin{aligned}
& x_0^n I_C^{(g)} \left(X_{K,L} \leftarrow \prod_{K' \subset K} x_{L_{K'}}^{(-1)^{|K|-|K'|}} \right) \\
&= x_0^n \sum_{u_1, \dots, u_g \in C} \prod_{1 \leq p \leq g} \prod_{K \in \binom{[g]}{p}} \prod_{L \in (\mathfrak{X}^*)^p} \left(\prod_{K' \subset K} x_{L_{K'}}^{(-1)^{|K|-|K'|}} \right)^{n_L(u_{K_1}, \dots, u_{K_p})} \\
&= x_0^n \sum_{u_1, \dots, u_g \in C} \prod_{\substack{a \in \mathfrak{X}^g, \\ a \neq 0}} \left(\prod_{\substack{K \subset \text{Vsupp}(a), \\ K \neq \emptyset}} \prod_{K' \subset K} x_{a_{K'}}^{(-1)^{|K|-|K'|}} \right)^{n_a(u_1, \dots, u_g)} \\
&= \sum_{u_1, \dots, u_g \in C} x_0^{n_0(u_1, \dots, u_g)} \prod_{\substack{a \in \mathfrak{X}^g, \\ a \neq 0}} \left(x_0 \prod_{\substack{K \subset \text{Vsupp}(a), \\ K \neq \emptyset}} \prod_{K' \subset K} x_{a_{K'}}^{(-1)^{|K|-|K'|}} \right)^{n_a(u_1, \dots, u_g)} \\
&= \sum_{u_1, \dots, u_g \in C} x_0^{n_0(u_1, \dots, u_g)} \prod_{\substack{a \in \mathfrak{X}^g, \\ a \neq 0}} x_a^{n_a(u_1, \dots, u_g)} \\
&= W_C^{(g)}.
\end{aligned}$$

Hence we complete the proof. \square

Example 3.1. Applying Theorem 3.1 in Example 1.1, we have

$$\begin{aligned}
& W_{C_2}^{(2)}(x_{(0,0)} \leftarrow 1, x_{(0,1)} \leftarrow X_{(2),(1)}, x_{(0,2)} \leftarrow X_{(2),(2)}, x_{(1,0)} \leftarrow X_{(1),(1)}, \\
& x_{(1,1)} \leftarrow X_{(1),(1)} X_{(2),(1)} X_{(1,2),(1,1)}, x_{(1,2)} \leftarrow X_{(1),(1)} X_{(2),(2)} X_{(1,2),(1,2)}, \\
& x_{(2,0)} \leftarrow X_{(1),(2)}, x_{(2,1)} \leftarrow X_{(1),(2)} X_{(2),(1)} X_{(1,2),(2,1)}, \\
& x_{(2,2)} \leftarrow X_{(1),(2)} X_{(2),(2)} X_{(1,2),(2,2)}) \\
&= I_{C_2}^{(2)}(X_{(1),(1)}, X_{(1),(2)}, X_{(2),(1)}, X_{(2),(2)}, \\
& X_{(1,2),(1,1)}, X_{(1,2),(1,2)}, X_{(1,2),(2,1)}, X_{(1,2),(2,2)}).
\end{aligned}$$

$$\begin{aligned}
& x_{(0,0)}^2 I_{C_2}^{(2)}(X_{(1),(1)} \leftarrow \frac{x_{(1,0)}}{x_{(0,0)}}, X_{(1),(2)} \leftarrow \frac{x_{(2,0)}}{x_{(0,0)}}, X_{(2),(1)} \leftarrow \frac{x_{(0,1)}}{x_{(0,0)}}, \\
& X_{(2),(2)} \leftarrow \frac{x_{(0,2)}}{x_{(0,0)}}, X_{(1,2),(1,1)} \leftarrow \frac{x_{(0,0)}x_{(1,1)}}{x_{(1,0)}x_{(0,1)}}, X_{(1,2),(1,2)} \leftarrow \frac{x_{(0,0)}x_{(1,2)}}{x_{(1,0)}x_{(0,2)}}, \\
& X_{(1,2),(2,1)} \leftarrow \frac{x_{(0,0)}x_{(2,1)}}{x_{(2,0)}x_{(0,1)}}, X_{(1,2),(2,2)} \leftarrow \frac{x_{(0,0)}x_{(2,2)}}{x_{(2,0)}x_{(0,2)}} \\
& = W_{C_2}^{(2)}(x_{(0,0)}, x_{(0,1)}, x_{(0,2)}, x_{(1,0)}, x_{(1,1)}, x_{(1,2)}, x_{(2,0)}, x_{(2,1)}, x_{(2,2)}).
\end{aligned}$$

3.2. Homogeneous and inhomogeneous Jacobi polynomials. In this section, we give a relation between homogeneous and inhomogeneous Jacobi polynomials.

Theorem 3.2. *Let C be an \mathfrak{R} -linear code of length n . Then*

$$\begin{aligned}
& \mathfrak{Jac}_{C,v}^{(g)}(\{y_a\}_{a \in \mathfrak{R}^{g+1}}) \\
& = (y_0)^n \prod_{\ell \in \mathfrak{R}^*} \left(\frac{y_{(0,\dots,0,\ell)}}{y_0} \right)^{\text{wt}_\ell(v)} \\
& \quad \text{Jac}_{C,v}^{(g)}(X_{(k),(\ell)} \leftarrow \left\{ \frac{y_{L(k)}}{y_0} \right\}, X_{K,L} \leftarrow \prod_{K' \subset K} y_{L_{K'}}^{(-1)^{|K|-|K'|}}),
\end{aligned}$$

where $L_{(k)}$ denotes a vector with length $g+1$ such that $L_{(k),i} = \ell$ if $i = k$, $L_{(k),i} = 0$ if $i \neq k$, and $K = (K_1, \dots, K_p) \in \binom{[g+1]}{p}$ for $2 \leq p \leq g+1$, and $K' = (K_{m_1}, \dots, K_{m_r})$ for $1 \leq r \leq p$ and $1 \leq m_1 \leq \dots \leq m_r \leq p$, and $L_{K'}$ is the length $g+1$ vector such that $L_{K',i} = L_j$ for $i = K_{m_j} \in K'$, where $1 \leq j \leq r$, $L_{K',i} = 0$ for $i \notin K'$, and $L_\emptyset = 0$.

Proof. From the right-hand side and using Lemma 2.1, we have

$$\begin{aligned}
& (y_0)^n \prod_{\ell \in \mathfrak{R}^*} \left(\frac{y_{(0, \dots, 0, \ell)}}{y_0} \right)^{\text{wt}_\ell(v)} \text{Jac}_{C,v}^{(g)}(X_{(k),(\ell)} \leftarrow \left\{ \frac{y_{L(k)}}{y_0} \right\}, \\
& \qquad \qquad \qquad X_{K,L} \leftarrow \prod_{K' \subset K} y_{L_{K'}}^{(-1)^{|K|-|K'|}} \\
& = (y_0)^n \sum_{u_1, \dots, u_g \in C} \left(\prod_{k \in [g]} \prod_{\ell \in \mathfrak{R}^*} \left\{ \frac{y_{L(k)}}{y_0} \right\}^{n_\ell(u_k)} \right) \\
& \qquad \prod_{2 \leq p \leq g+1} \prod_{\substack{K \in \binom{[g+1]}{p}, \\ u_{g+1} = v}} \prod_{L \in (\mathfrak{R}^*)^p} \left(\prod_{K' \subset K} y_{L_{K'}}^{(-1)^{|K|-|K'|}} \right)^{n_L(u_{K_1}, \dots, u_{K_p})} \\
& = \sum_{u_1, \dots, u_g \in C} y_0^{n_0(u_1, \dots, u_g, v)} \prod_{\substack{a \in \mathfrak{R}^{g+1}, \\ a \neq 0}} \left(y_0 \prod_{\substack{K \subset \text{Vsupp}(a), \\ K \neq \emptyset, \\ v = u_{g+1}}} \prod_{K' \subset K} y_{a_{K'}}^{(-1)^{|K|-|K'|}} \right)^{n_a(u_1, \dots, u_g, v)} \\
& = \sum_{u_1, \dots, u_g \in C} y_0^{n_0(u_1, \dots, u_g, v)} \prod_{\substack{a \in \mathfrak{R}^{g+1}, \\ a \neq 0}} y_a^{n_a(u_1, \dots, u_g, v)} \\
& = \mathfrak{Jac}_{C,v}^{(g)}(\{y_a\}_{a \in \mathfrak{R}^{g+1}}).
\end{aligned}$$

This completes the proof. \square

3.3. Intersection enumerators and Jacobi polynomials. In this section, we give a relation between intersection enumerators and Jacobi polynomials.

Theorem 3.3. *Let C be an \mathfrak{R} -linear code of length n . Let $s := |\mathfrak{R}^*|$. Denote the elements of \mathfrak{R}^* by ℓ_1, \dots, ℓ_s . Then the following hold.*

$$I_C^{(g+1)} = \sum_{r=0}^n \sum_{\substack{r_1, \dots, r_s \in \mathbb{Z}_{\geq 0} \\ r_1 + \dots + r_s = r \\ \ell_1, \dots, \ell_s \in \mathfrak{R}^*}} \left(\sum_{\substack{v \in C, \\ \text{wt}_{\ell_i}(v) = r_i}} \text{Jac}_{C,v}^{(g)} \right) X_{(g+1), \ell_1}^{r_1} \cdots X_{(g+1), \ell_s}^{r_s}.$$

Proof. Let C be an \mathfrak{R} -linear code of length n . Then the $(g+1)$ -th intersection enumerator for a code C can be written as:

$$\begin{aligned}
& I_C^{(g+1)}(\{X_{K,L}\}_{1 \leq p \leq g+1, K \in \binom{[g+1]}{p}, L \in (\mathfrak{R}^*)^p}) \\
&= \sum_{u_1, \dots, u_{g+1} \in C} \prod_{1 \leq p \leq g+1} \prod_{K \in \binom{[g+1]}{p}} \prod_{L \in (\mathfrak{R}^*)^p} X_{K,L}^{nL(u_{K_1}, \dots, u_{K_p})} \\
&= \sum_{u_{g+1} \in C} \left\{ \sum_{u_1, \dots, u_g \in C} \left(\prod_{k \in [g]} \prod_{\ell \in \mathfrak{R}^*} X_{(k),(\ell)}^{n\ell(u_k)} \right) \right. \\
&\quad \left. \prod_{2 \leq p \leq g+1} \prod_{K \in \binom{[g+1]}{p}} \prod_{L \in (\mathfrak{R}^*)^p} X_{K,L}^{nL(u_{K_1}, \dots, u_{K_p})} \right\} \prod_{L \in \mathfrak{R}^*} X_{(g+1),L}^{nL(u_{g+1})} \\
&= \sum_{u_{g+1} \in C} \text{Jac}_{C, u_{g+1}}^{(g)}(\{X_{(k),(\ell)}\}_{k \in [g], \ell \in \mathfrak{R}^*}, \\
&\quad \{X_{K,L}\}_{2 \leq p \leq g+1, K \in \binom{[g+1]}{p}, L \in (\mathfrak{R}^*)^p}) \prod_{L \in \mathfrak{R}^*} X_{(g+1),L}^{nL(u_{g+1})} \\
&= \sum_{r=0}^n \sum_{\substack{r_1, \dots, r_s \in \mathbb{Z}_{\geq 0} \\ r_1 + \dots + r_s = r \\ \ell_1, \dots, \ell_s \in \mathfrak{R}^*}} \sum_{\substack{v \in C, \\ \text{wt}_{\ell_i}(v) = r_i}} \text{Jac}_{C,v}^{(g)}(\{X_{(k),(\ell)}\}_{k \in [g], \ell \in \mathfrak{R}^*}, \\
&\quad \{X_{K,L}\}_{2 \leq p \leq g+1, K \in \binom{[g+1]}{p}, L \in (\mathfrak{R}^*)^p}) X_{(g+1),\ell_1}^{r_1} \cdots X_{(g+1),\ell_s}^{r_s}.
\end{aligned}$$

Hence the proof is completed. \square

4. MACWILLIAMS TYPE IDENTITIES

In this section, we give two MacWilliams type identities for g -th homogeneous and g -th inhomogeneous Jacobi polynomials. We recall [8, 13] to introduce some fixed characters over \mathfrak{R} .

Let $\mathfrak{R} = \mathbb{F}_q$, where $q = p^f$ for some prime number p . A *character* χ of \mathbb{F}_q is a homomorphism from the additive group \mathbb{F}_q to the multiplicative group of non-zero complex numbers. Now let $F(x)$ be a primitive irreducible polynomial of degree f over \mathbb{F}_p and let λ be a root of $F(x)$. Then any element $\alpha \in \mathbb{F}_q$ has a unique representation as:

$$(1) \quad \alpha = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \cdots + \alpha_{f-1} \lambda^{f-1},$$

where $\alpha_i \in \mathbb{F}_p$. We define $\chi(\alpha) := \zeta_p^{\alpha_0}$, where ζ_p is the primitive complex p -th root of unity $\zeta_p = e^{2\pi i/p}$, and α_0 is given by Equation (1).

Again if $\mathfrak{R} = \mathbb{Z}_k$, then for $\alpha \in \mathbb{Z}_k$, we define χ as $\chi(\alpha) := \zeta_k^\alpha$, where ζ_k is the primitive complex k -th root of unity $\zeta_k = e^{2\pi i/k}$.

For any $\alpha \in \mathfrak{R}$, we have the following property:

$$\sum_{\beta \in \mathfrak{R}} \chi(\alpha\beta) := \begin{cases} |\mathfrak{R}| & \text{if } \alpha = 0, \\ 0 & \text{if } \alpha \neq 0. \end{cases}$$

4.1. Homogeneous Jacobi polynomials. The MacWilliams type identity for the g -fold complete joint Jacobi polynomials were discussed in [6, Theorem 5.1]. Then it is easy to give the MacWilliams type identity for the g -th homogeneous Jacobi polynomials as follows.

Theorem 4.1. *Let C be an \mathfrak{R} -linear code of length n . Then we have*

$$\begin{aligned} & \mathfrak{J}ac_{C^\perp, v}^{(g)}(\{y_a\}_{a \in \mathfrak{R}^{g+1}}) \\ &= \frac{1}{|C|^g} \mathfrak{J}ac_{C, v}^{(g)} \left(\left\{ \sum_{\substack{b_1, \dots, b_g \in \mathfrak{R} \\ b_{g+1} = a_{g+1}}} \chi \left(\sum_{i=1}^g a_i b_i \right) y_{(b_1, \dots, b_g, b_{g+1})} \right\}_{a \in \mathfrak{R}^{g+1}} \right). \end{aligned}$$

Proof. The proof is similar to the proof of [17, Theorem 12]. \square

4.2. Inhomogeneous Jacobi polynomials. In this section, we give a MacWilliams type identity for inhomogeneous Jacobi polynomials.

Theorem 4.2. *Let C be an \mathfrak{R} -linear code of length n . Then we have the following MacWilliams type relation:*

$$\begin{aligned} & Jac_{C^\perp, v}^{(g)}(\{X_{(k), (\ell)}\}_{k \in [g], \ell \in \mathfrak{R}^*}, \{X_{K, L}\}_{2 \leq p \leq g+1, K \in \binom{[g+1]}{p}, L \in (\mathfrak{R}^*)^p}) \\ &= (y_0)^n \prod_{\ell \in \mathfrak{R}^*} \left(\frac{y_{(0, \dots, 0, \ell)}}{y_0} \right)^{\text{wt}_\ell(v)} \\ & \quad Jac_{C, v}^{(g)}(X_{(k), (\ell)} \leftarrow \left\{ \frac{y_{L_{(k)}}}{y_0} \right\}, X_{K, L} \leftarrow \prod_{K' \subset K} y_{L_{K'}}^{(-1)^{|K| - |K'|}}), \end{aligned}$$

where $L_{(k)}$ denotes a vector with length $g+1$ such that $L_{(k), i} = \ell$ if $i = k$, $L_{(k), i} = 0$ if $i \neq k$, and $K = (K_1, \dots, K_p) \in \binom{[g+1]}{p}$ with $K_1 < \dots < K_p$ for $2 \leq p \leq g+1$, and $K' = (K_{m_1}, \dots, K_{m_r})$ for $1 \leq r \leq p$ and $1 \leq m_1 \leq \dots \leq m_r \leq p$, and $L_{K'}$ is the length $g+1$ vector such that $L_{K', i} = L_j$ for $i = K_{m_j} \in K'$, where $1 \leq j \leq r$, $L_{K', i} = 0$ for $i \notin K'$, and $L_\emptyset = 0$, and for $a \in \mathfrak{R}^{g+1}$,

$$y_a = \sum_{\substack{b_1, \dots, b_g \in \mathfrak{R} \\ b_{g+1} = a_{g+1}}} \chi \left(\sum_{i=1}^g a_i b_i \right) \prod_{1 \leq p \leq g+1} \prod_{K \in \binom{[g+1]}{p}} X_{K, B_K},$$

where B_K is the length $|K|$ vector indexed by K such that $B_K = (b_{i_1}, \dots, b_{i_{|K|}})$ with $i_1 < \dots < i_{|K|}$, and $X_{\emptyset, B_\emptyset} = 1$, and $X_{(g+1), (\ell)} = 1$ for all $\ell \in \mathfrak{R}$.

Proof. For any $w \in \mathfrak{R}^n$, we define

$$\delta_{C^\perp}(w) := \begin{cases} 1 & \text{if } w \in C^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following identity

$$\delta_{C^\perp}(w) = \frac{1}{|C|} \sum_{u \in C} \chi(u \cdot w).$$

Now

$$\begin{aligned} & \text{Jac}_{C^\perp, v}^{(g)}(\{X_{(k), (\ell)}\}_{k \in [g], \ell \in \mathfrak{R}^*}, \{X_{K, L}\}_{2 \leq p \leq g+1, K \in \binom{[g+1]}{p}, L \in (\mathfrak{R}^*)^p}) \\ &= \sum_{u_1, \dots, u_g \in C^\perp} \left(\prod_{k \in [g]} \prod_{\ell \in \mathfrak{R}^*} X_{(k), (\ell)}^{n_\ell(u_k)} \right) \\ & \quad \left(\prod_{2 \leq p \leq g+1} \prod_{\substack{K \in \binom{[g+1]}{p} \\ u_{g+1} = v}} \prod_{L \in (\mathfrak{R}^*)^p} X_{K, L}^{n_L(u_{K_1}, \dots, u_{K_p})} \right) \\ &= \sum_{u_1, \dots, u_g \in C^\perp} \prod_{1 \leq p \leq g+1} \prod_{\substack{K \in \binom{[g+1]}{p} \\ K \neq (g+1) \\ u_{g+1} = v}} \prod_{L \in (\mathfrak{R}^*)^p} X_{K, L}^{n_L(u_{K_1}, \dots, u_{K_p})} \\ &= \sum_{w_1, \dots, w_g \in \mathfrak{R}^n} \prod_{i=1}^g \delta_{C^\perp}(w_i) \prod_{1 \leq p \leq g+1} \prod_{\substack{K \in \binom{[g+1]}{p} \\ K \neq (g+1) \\ w_{g+1} = v}} \prod_{L \in (\mathfrak{R}^*)^p} X_{K, L}^{n_L(w_{K_1}, \dots, w_{K_p})} \\ &= \sum_{w_1, \dots, w_g \in \mathfrak{R}^n} \prod_{i=1}^g \left(\frac{1}{|C|} \sum_{u_i \in C} \chi(u_i \cdot w_i) \right) \\ & \quad \prod_{1 \leq p \leq g+1} \prod_{\substack{K \in \binom{[g+1]}{p} \\ K \neq (g+1) \\ w_{g+1} = v}} \prod_{L \in (\mathfrak{R}^*)^p} X_{K, L}^{n_L(w_{K_1}, \dots, w_{K_p})} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{|C|^g} \sum_{\substack{u_1, \dots, u_g \in C \\ w_1, \dots, w_g \in \mathfrak{A}^n \\ w_{g+1} = v}} \chi\left(\sum_{i=1}^g u_{i,1} w_{i,1} + \dots + u_{i,n} w_{i,n}\right) \\
&\quad \prod_{1 \leq p \leq g+1} \prod_{\substack{K \in \binom{[g+1]}{p} \\ K \neq (g+1)}} \prod_{L \in (\mathfrak{A}^*)^p} X_{K,L}^{n_L(w_{K_1}, \dots, w_{K_p})} \\
&= \frac{1}{|C|^g} \sum_{u_1, \dots, u_g \in C} \prod_{1 \leq i \leq n} \left\{ \sum_{w_{1,i}, \dots, w_{g,i} \in \mathfrak{A}} \chi(u_{1,i} w_{1,i} + \dots + u_{g,i} w_{g,i}) \right. \\
&\quad \left. \left(\prod_{1 \leq p \leq g+1} \prod_{\substack{K \in \binom{[g+1]}{p} \\ K \neq (g+1) \\ w_{g+1} = v}} X_{K, (w_{K_1, i}, \dots, w_{K_p, i})} \right) \right\} \\
&= \frac{1}{|C|^g} \sum_{u_1, \dots, u_g \in C} \prod_{a \in \mathfrak{A}^{g+1}} \left\{ \sum_{\substack{b_1, \dots, b_g \in \mathfrak{A} \\ b_{g+1} = a_{g+1}}} \chi(a_1 b_1 + \dots + a_g b_g) \right. \\
&\quad \left. \left(\prod_{1 \leq p \leq g+1} \prod_{\substack{K \in \binom{[g+1]}{p} \\ K \neq (g+1)}} X_{K, B_K} \right) \right\}^{n_a(u_1, \dots, u_g, v)} \\
&= \frac{1}{|C|^g} \sum_{u_1, \dots, u_g \in C} \prod_{a \in \mathfrak{A}^{g+1}} y_a^{n_a(u_1, \dots, u_g, v)} \\
&= \frac{1}{|C|^g} \sum_{u_1, \dots, u_g \in C} y_0^n \\
&\quad \prod_{1 \leq p \leq g+1} \prod_{\substack{K \in \binom{[g+1]}{p} \\ v = u_{g+1}}} \prod_{L \in (\mathfrak{A}^*)^p} \left\{ \prod_{K' \subset K} y_{L_{K'}}^{(-1)^{|K| - |K'|}} \right\}^{n_L(u_{K_1}, \dots, u_{K_p})}
\end{aligned}$$

(by Lemma 2.1)

$$\begin{aligned}
&= \frac{1}{|C|^g} y_0^n \prod_{\ell \in \mathfrak{X}^*} \left(\frac{y_{(0, \dots, 0, \ell)}}{y_0} \right)^{\text{wt}_\ell(v)} \\
&\quad \sum_{u_1, \dots, u_g \in C} \prod_{1 \leq p \leq g+1} \prod_{\substack{K \in \binom{[g+1]}{p} \\ K \neq (g+1) \\ v = u_{g+1}}} \prod_{L \in (\mathfrak{X}^*)^p} \left\{ \prod_{K' \subset K} y_{L_{K'}}^{(-1)^{|K|-|K'|}} \right\}^{n_L(u_{K_1}, \dots, u_{K_p})}.
\end{aligned}$$

This completes the proof. \square

We will discuss a generalization of the Broué–Enguehard map [5] and Bannai–Ozeki map [2] to the case Jacobi polynomials of genus g in our subsequent papers. Bonnecaze et al. [4] gave a connection between design theory and Jacobi polynomials. A generalization of the connection to the case genus g Jacobi polynomials with multiple reference vectors was given in [7]. This gives rise to a natural question: is it possible to generalize the results of this paper to the case genus g Jacobi polynomials with multiple reference vectors g ? In our future work, we shall investigate this question.

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