Type II Codes, Even Unimodular Lattices and Invariant Rings

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Abstract

In this paper, we study self-dual codes over the ring $\mathbb{Z}_{2k}$ of the integers modulo $2k$ with relationships to even unimodular lattices, modular forms, and invariant rings of

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finite groups. We introduce Type II codes over $\mathbb{Z}_{2k}$ which are closely related to even unimodular lattices, as a remarkable class of self-dual codes and a generalization of binary Type II codes. A construction of even unimodular lattices is given using Type II codes. Several examples of Type II codes are given, in particular the first extremal Type II code over $\mathbb{Z}_6$ of length 24 is constructed, which gives a new construction of the Leech lattice. The complete and symmetrized weight enumerators in genus g of codes over $\mathbb{Z}_{2k}$ are introduced, and the MacWilliams identities for these weight enumerators are given. We investigate the groups which fix these weight enumerators of Type II codes over $\mathbb{Z}_{2k}$ and we give the Molien series of the invariant rings of the groups for small cases. We show that modular forms are constructed from complete and symmetrized weight enumerators of Type II codes. Shadow codes over $\mathbb{Z}_{2k}$ are also introduced.

Index Term: Codes over $\mathbb{Z}_{2k}$, Type II codes, even unimodular lattices, invariant rings.

1 Introduction

Recently there has been interest in self-dual codes over finite rings, especially, the ring $\mathbb{Z}_4$ of integers modulo 4. The best known nonlinear binary codes such as the Nordstrom-Robinson, Kerdock, Preparata, Goethals and Delsarte-Goethals codes contain more codewords than any known linear codes with the same minimum distance. A simple relationship between these nonlinear binary codes and self-dual codes over $\mathbb{Z}_4$ was discovered by Hammons, Kumar, Calderbank, Sloane and Solé [14]. Moreover, similarly to binary self-dual codes it was shown that self-dual codes over $\mathbb{Z}_4$ are closely related to unimodular lattices via Construction A_4 [2], in particular, any extremal Type II code of length 24 gives an alternative construction of the Leech lattice. The notion of Type II codes over $\mathbb{Z}_4$ was introduced in [3]. More recently as simple generalizations, cyclic self-dual codes over $\mathbb{Z}_{2^m}$, especially the lifted Hamming and Golay codes have been investigated in [4] and Type II codes over $\mathbb{Z}_{2^m}$ have been studied in [9]. It is natural to consider the ring $\mathbb{Z}_{2^m}$ for cyclic codes since the Hensel lift plays an important role, however there is no need to restrict the order of rings when considering an application to unimodular lattices. The Chinese remainder theorem is a useful tool to investigate codes over $\mathbb{Z}_k$ [11].

In this paper, we study self-dual codes over $\mathbb{Z}_{2k}$. In Section 2, we give definitions and some basic facts. We also introduce Type II codes over $\mathbb{Z}_{2k}$ as a remarkable class of self-dual codes then we show such codes are closely related to even unimodular lattices in Section 3. This relationship provides a number of properties of Type II codes. In Section 4, several examples of extremal self-dual codes are constructed giving construction methods. For example, the first extremal Type II code over $\mathbb{Z}_6$ of length 24 is constructed, which gives a new construction of the Leech lattice. Section 5 introduces the complete and symmetrized weight enumerators in genus g of codes over $\mathbb{Z}_{2k}$. The MacWilliams identities for those weight enumerators are provided. We also investigate the groups which fix weight enumerators of Type II codes over $\mathbb{Z}_{2k}$. Section 6 investigates shadow codes of Type I codes over $\mathbb{Z}_{2k}$. In Section 7, modular forms are constructed from weight enumerators of Type II codes. In Section 8, we give the Molien series for the invariant rings corresponding to the complete and symmetrized weight enumerators in genus g of Type II codes over $\mathbb{Z}_{2k}$ for small k and g.

2 Definitions and Basic Facts

In this section, we first give the definitions used throughout this paper. Then we introduce Type II codes. Some basic properties of the Euclidean weight are also given.

A *linear* code C of length n over $\mathbb{Z}_{2k}$ is an additive subgroup of $\mathbb{Z}_{2k}^n$. A *nonlinear* code C of length n is simply a subset of $\mathbb{Z}_{2k}^n$. In this paper, we consider only linear codes. An element of C is called a codeword of C. A *generator* matrix of C is a matrix whose rows generate C. The *Hamming weight* $wt_H(x)$ of a vector x in $\mathbb{Z}_{2k}^n$ is the number of non-zero components. The *Euclidean weight* $wt_E(x)$ of a vector $x = (x_1, x_2, \ldots, x_n)$ is $\sum_{i=1}^n \min\{x_i^2, (2k - x_i)^2\}$. The *Lee weight* $wt_L(x)$ of a vector x is $\sum_{i=1}^n \min\{|x_i|, |2k - x_i|\}$. The Hamming, Lee and Euclidean distances $d_H(x, y)$, $d_L(x, y)$ and $d_E(x, y)$ between two vectors x and y are $wt_H(x - y)$, $wt_L(x - y)$ and $wt_E(x - y)$, respectively. The minimum Hamming, Lee and Euclidean weights, d_H, d_L and d_E, of C are the smallest Hamming, Lee and Euclidean weights among all non-zero codewords of C, respectively.

We define the inner product of x and y in $\mathbb{Z}_{2k}^n$ by $< x, y >= x_1y_1 + \cdots + x_ny_n \pmod{2k}$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. The *dual code* $C^\perp$ of C is defined as $C^\perp = \{x \in \mathbb{Z}_{2k}^n \mid < x, y >= 0 \text{ for all } y \in C\}$. C is *self-orthogonal* if $C \subseteq C^\perp$ and C is *self-dual* if $C = C^\perp$. We define a *Type II* code over $\mathbb{Z}_{2k}$ as a self-dual code with Euclidean weights divisible by $4k$. For $k = 1$, this is the standard definition of binary Type II codes. For $k = 2$, the original definition given in [3] requires that the code contains the all-one vector as well, however recently it has been shown in [16] that such a Type II code in terms of [3] is equivalent to a Type II code by our definition. Self-dual codes which are not Type II are said to be *Type I.*

For some applications, there is often no need to distinguish between $+1$ and -1 components of codewords, and we say that two codes are *equivalent* if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates. Codes differing by only a permutation of coordinates are called *permutation-equivalent.*

The *complete weight enumerator* (cwe for short) of a code C over $\mathbb{Z}_{2k}$ is defined as

$$cwe_C(x_0, x_1, \ldots, x_{2k-1}) = \sum_{c \in C} x_0^{n_0(c)} x_1^{n_1(c)} \cdots x_{2k-2}^{n_{2k-2}(c)} x_{2k-1}^{n_{2k-1}(c)},$$

where $n_i(c)$ is the number of i components of c, respectively. Permutation-equivalent codes have the identical cwe's but equivalent codes may have different cwe's. The appropriate weight enumerator for equivalent codes is the *symmetrized weight enumerator* (swe for short) defined as

$$swe_C(x_0, x_1, \ldots, x_k) = \sum_{c \in C} x_0^{n'_0(c)} x_1^{n'_1(c)} \cdots x_{k-1}^{n'_{k-1}(c)} x_k^{n'_k(c)},$$

where $n'_0(x), n'_1(x), \ldots, n'_{k-1}(c), n'_k(c)$ are the numbers of $0, \pm 1, \ldots, \pm k-1, k$ components of c, respectively.

Let $\{q_1, q_2, \ldots, q_r\}$ be the set of integers less than $2k$ that divide $2k$, and arranged so that $q_i < q_j$ for $i < j$. Note that this implies $q_1 = 1$. Any code over $\mathbb{Z}_{2k}$ is permutation-equivalent to a code with generator matrix of the form

$$(1) \qquad \begin{pmatrix} q_1 I_{k_1} & A_{1,2} & A_{1,3} & A_{1,4} & \cdots & \cdots & A_{1,r+1} \\ 0 & q_2 I_{k_2} & q_2 A_{2,3} & q_2 A_{2,4} & \cdots & \cdots & q_2 A_{2,r+1} \\ 0 & 0 & q_3 I_{k_3} & q_3 A_{3,4} & \cdots & \cdots & q_3 A_{3,r+1} \\ \vdots & \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & q_r I_{k_r} & q_r A_{r,r+1} \end{pmatrix},$$

where $A_{i,j}$ are binary matrices for $i > 1$. A code of this form is said to be of *rank* $\{q_1{}^{k_1}, q_2{}^{k_2}, q_3{}^{k_3}, \ldots, q_r{}^{k_r}\}$ and it has $\prod_{j=1}^{r} (\frac{s}{q_j})^{k_j}$ codewords.

We now give basic properties of Euclidean weights over $\mathbb{Z}_{2k}$.

Lemma 2.1 *Let x be a vector in $\mathbb{Z}_{2k}^n$. Then $wt_E(x) \equiv < x, x > \pmod{4k}$.*

Proof. Follows from the definition of the Euclidean weight. □

Lemma 2.2 *Let M be a generator matrix of a code C. Suppose that the rows of M are vectors in $\mathbb{Z}_{2k}^n$ with Euclidean weight a multiple of $4k$ with any two rows orthogonal. Then C is a self-orthogonal code with all Euclidean weights a multiple of $4k$.*

Proof. Let r_i be the i-th row of M. By Lemma 2.1,

$$(2) \qquad wt_E(x+y) \equiv wt_E(x) + wt_E(y) + 2 < x, y > \pmod{4k}.$$

This shows the lemma. □

By the above lemma, it is sufficient to obtain the Euclidean weights of all the rows in a generator matrix of a code C when we check if C is Type II.

We now introduce the notion of shadows for Type I codes over $\mathbb{Z}_{2k}$. We first define a specific coset of a Type I code C over $\mathbb{Z}_{2k}$ in order to define the shadows. The *$4k$-weight subcode* C_0 of a Type I code C is the set of codewords of C of Euclidean weights divisible by $4k$.

Lemma 2.3 *The subcode C_0 is a linear subcode of index 2 in C.*

Proof. By (2), the sum of two codewords in C_0 is in C_0. Every vector in C has a Euclidean weight divisible by $2k$. By (2) we see that $C_2 = C - C_0$ is of the form $x + C_0$ where x is any codeword of C of Euclidean weight congruent to $2k \pmod{4k}$ and that translation by x is a one to one map from C_0 onto C_2. □

Define the *shadow* of C as $S = C_0^\perp - C$. The shadows for binary Type I codes were introduced by Conway and Sloane [6]. This notion was applied to Type I codes over $\mathbb{Z}_4$ in [10]. Unlike the binary case, $C_0^\perp/C_0$ is not necessarily isomorphic to the Klein 4-group; it may be isomorphic to either the Klein 4-group or the cyclic group of order 4.

3 Even Unimodular Lattices and Type II Codes

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space with the inner product $[x,y] = x_1y_1 + x_2y_2 + \cdots + x_ny_x$ for $x = (x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$. An n-dimensional lattice Λ in $\mathbb{R}^n$ is a free $\mathbb{Z}$-module spanned by n linearly independent vectors $v_1, \ldots, v_n$. An n by n matrix whose rows are the vectors $v_1, \ldots, v_n$ is called a generator matrix G of Λ. The fundamental volume $V(\Lambda)$ of Λ is $|\det G|$. For a sublattice $\Lambda' \subset \Lambda$, it holds that $V(\Lambda') = V(\Lambda)|\Lambda/\Lambda'|$. The *dual* lattice Λ^* is given by $\Lambda^* = \{x \in \mathbb{R}^n \,|\, [x,a] \in \mathbb{Z} \text{ for all } a \in \Lambda\}$. A lattice Λ is *integral* if $\Lambda \subseteq \Lambda^*$. An integral lattice with $\det \Lambda = 1$ (or $\Lambda = \Lambda^*$) is called *unimodular*. If the norm $[x,x]$ is an even integer for all $x \in \Lambda$, then Λ is called *even*. Unimodular lattices which are not even are called *odd*. The minimum norm of Λ is the smallest norm among all nonzero vectors of Λ.

Applying Construction A in [7] to Type II codes over $\mathbb{Z}_{2k}$, we have the following construction of even unimodular lattices. Let ρ be a map from $\mathbb{Z}_{2k}$ to $\mathbb{Z}$ sending $0, 1, \ldots, k$ to $0, 1, \ldots, k$ and $k+1, \ldots, 2k-1$ to $1-k, \ldots, -1$, respectively.

Theorem 3.1 *If C is a self-dual code of length n over $\mathbb{Z}_{2k}$, then the lattice*

$$\Lambda(C) = \frac{1}{\sqrt{2k}}\{\rho(C) + 2k\mathbb{Z}^n\},$$

is an n-dimensional unimodular lattice, where $\rho(C) = \{(\rho(c_1), \ldots, \rho(c_n)) \,|\, (c_1, \ldots, c_n) \in C\}$. The minimum norm is $\min\{2k, d_E/2k\}$ where d_E is the minimum Euclidean weight of C. Moreover, if C is Type II then the lattice $\Lambda(C)$ is an even unimodular lattice.

Proof. If $a_1, a_2 \in \Lambda(C)$ then $a_i = (c_i + 2kz_i)/\sqrt{2k}$ where $c_i \in \rho(C)$ and $z_i \in \mathbb{Z}^n$ for $i = 1$ and 2. Since C is self-dual, the inner product of a_1 and a_2 is

$$[a_1, a_2] = \frac{1}{2k}\{[c_1, c_2] + 2k[z_1, c_2] + 2k[c_1, z_2] + 4k^2[z_1, z_2]\} \in \mathbb{Z},$$

thus $\Lambda(C)$ is integral. In addition, if C is Type II then the Euclidean weights are divisible by $4k$. Then we have

$$[a_1, a_1] = \frac{1}{2k}\{[c_1, c_1] + 4k[z_1, z_1] + 4k^2[c_1, z_1]\} \in 2\mathbb{Z},$$

so that the lattice is even.

Consider the lattice $\sqrt{2k}\Lambda(C)$, then

$$2k\mathbb{Z}^n \subset \sqrt{2k}\Lambda(C) \subset \mathbb{Z}^n.$$

Since $V(2k\mathbb{Z}^n) = (2k)^n$ and $|\sqrt{2k}\Lambda(C)/2k\mathbb{Z}^n| = (2k)^{n/2}$, we have $V(\sqrt{2k}\Lambda(C)) = (2k)^{n/2}$. Then $V(\Lambda(C)) = 1$ and $\Lambda(C)$ is unimodular.

It is easy to see that $[a_i, a_i] \geq [c_i/\sqrt{2k}, c_i/\sqrt{2k}]$ where $a_i = (c_i + 2kz_i)/\sqrt{2k}$. Thus the minimum norm is $\min\{2k, d_E/2k\}$. □

Theorem 3.1 provides much information on Type II codes over $\mathbb{Z}_{2k}$. For example, the following corollary characterizes divisible self-dual codes over $\mathbb{Z}_{2k}$ in terms of their Euclidean weights.

Corollary 3.2 *Suppose that C is a self-dual code over $\mathbb{Z}_{2k}$ which has the property that every Euclidean weight is a multiple of a positive integer. Then the largest positive integer c is either $2k$ or $4k$.*

Proof. If a unimodular lattice has the property that every norm is a multiple of some positive integer d then d is either 1 or 2 (cf. [19]). If C is self-dual then $\Lambda(C)$ is unimodular. Thus c must be either $2k$ or $4k$. □

Remark. Type I and Type II codes correspond to odd and even unimodular lattices, respectively.

Moreover, Theorem 3.1 gives a restriction of the length of a Type II code.

Corollary 3.3 *If there exists a Type II code C of length n over $\mathbb{Z}_{2k}$, then n is a multiple of eight.*

Proof. An even unimodular lattice of dimension n can be constructed from C by Theorem 3.1. Even unimodular lattices exist if and only if the dimension is a multiple of eight. Thus n must be a multiple of eight. □

Now let us consider the converse assertion of Corollary 3.3.

Proposition 3.4 *There exists a Type II code C of length n over $\mathbb{Z}_{2k}$ if and only if n is a multiple of eight.*

Proof. Consider the matrix

$$(\ I_4 \ , \ M_4 \ ),$$

where I_4 is the identity matrix of order 4 and

$$M_4 = \begin{pmatrix} a & b & c & d \\ b & -a & -d & c \\ c & d & -a & -b \\ d & -c & b & -a \end{pmatrix},$$

then $M_4 \cdot {}^tM_4 = (a^2 + b^2 + c^2 + d^2)I_4$ over $\mathbb{Z}$ where ${}^tA$ denotes the transpose matrix of a matrix A. From Lagrange's theorem on sums of squares, there are elements a, b, c, d of $\mathbb{Z}$ such that $1+a^2+b^2+c^2+d^2 = 4k$ for any k with $k > 0$. The integers a, b, c, d are necessarily less that or equal to $2k$ so there exists a, b, c, d of $\mathbb{Z}_{2k}$ such that $1 + a^2 + b^2 + c^2 + d^2 = 4k$ for $k > 0$. Therefore these elements a, b, c, d of $\mathbb{Z}_{2k}$ give that the matrix (I_4 , M_4) generates a Type II code of length 8 over $\mathbb{Z}_{2k}$ for any positive k. Note that Calderbank and Sloane [4] gave the lifted Hamming codes which are Type II codes of length 8 for $\mathbb{Z}_{2^m}$. □

The above Type II codes of length 8 give different constructions for the Gosset lattice E_8 which is the unique 8-dimensional even unimodular lattice.

We now investigate the minimum Euclidean weight of Type II codes over $\mathbb{Z}_{2k}$. The minimum norm μ of an n-dimensional even unimodular lattice is bounded by $\mu \le 2\lfloor \frac{n}{24} \rfloor + 2$ and even unimodular lattices with $\mu = 2\lfloor \frac{n}{24} \rfloor + 2$ are called *extremal* (cf. [7]). The minimum norm of the lattices constructed from Type II codes C gives directly an upper bound on the minimum Euclidean weight of C.

Corollary 3.5 *Let d_E be the minimum Euclidean weight of a Type II code of length $8n$ over $\mathbb{Z}_{2k}$. If $\lfloor \frac{n}{3} \rfloor \le k - 2$, then*

$$d_E \le 4k(\lfloor \frac{n}{3} \rfloor + 1). \tag{3}$$

Proof. Suppose that there exists a Type II code C with minimum Euclidean weight $d_E = 4k(\lfloor \frac{n}{3} \rfloor + 2)$. The minimum norm μ of the even unimodular lattice $\Lambda(C)$ constructed from C is $\min\{2k, 2\lfloor \frac{n}{3} \rfloor + 4\}$. From the assumption, $\mu = 2\lfloor \frac{n}{3} \rfloor + 4$, which is a contradiction. □

Remark. When $k = 1$ and 2, the above bound (3) holds without the assumption $\lfloor \frac{n}{3} \rfloor \le k-2$ (cf. [3], [18]). For $k = 1$ and 2, (3) is a bound for binary doubly-even self-dual codes and Type II codes over $\mathbb{Z}_4$. Thus the following conjecture is natural.

Conjecture 3.6 *The minimum Euclidean weight d_E is bounded by $d_E \le 4k(\lfloor \frac{n}{3} \rfloor + 1)$ for all $k \ge 1$.*

When $\lfloor \frac{n}{3} \rfloor \le k-2$, we say that Type II codes over $\mathbb{Z}_{2k}$ with $d_E = 4k(\lfloor \frac{n}{3} \rfloor + 1)$ are *extremal* for $k \ge 3$.

Recently Rains and Sloane [21] have proved that the minimum norm of μ of an n-dimensional unimodular lattice is bounded by $\mu \le 2\lfloor \frac{n}{24} \rfloor + 2$ unless $n = 23$ when $\mu \le 3$.

Corollary 3.7 *Let d_E be the minimum Euclidean weight of a Type I self-dual code of length n over $\mathbb{Z}_{2k}$. If $2\lfloor \frac{n}{24} \rfloor \le 2k-3$, then*

$$d_E \le \begin{cases} 4k(\lfloor \frac{n}{24} \rfloor + 1) & n \ne 23, \\ 6k & n = 23 \end{cases} \tag{4}$$

When $2\lfloor \frac{n}{24} \rfloor \le 2k-3$, Type I codes over $\mathbb{Z}_{2k}$ meeting the above bound (4) with equality are called *extremal*.

Remark. It is natural to define the Euclidean weights of the elements $0, \pm1, \pm2, \pm3, \cdots, \pm(k-1), \pm k$ of $\mathbb{Z}_{2k+1}$ as $0, 1, 4, 9, \cdots, (k-1)^2, k^2$, respectively. If C is a self-dual code over $\mathbb{Z}_{2k+1}$ then the lattice $\Lambda(C)$ in Theorem 3.1 is a unimodular lattice. However even if C is a self-dual code with all vectors having Euclidean weight a multiple of $4k+2$, then $\Lambda(C)$ is not always even. For example, the Euclidean weight of a vector (1123) over $\mathbb{Z}_5$ is 10 but the norm is 15. Moreover, the sum of two even vectors in $\mathbb{Z}_{2k+1}$ is not necessarily an even vector, for example the sum of (112) and itself in $\mathbb{Z}_3$ is (221) which is not even. Thus in this paper we consider Type II codes over $\mathbb{Z}_k$ for only even numbers k.

4 Extremal Self-Dual Codes

4.1 Extremal Type II Codes over $\mathbb{Z}_6$ and $\mathbb{Z}_8$

The most remarkable length for extremal Type II codes is 24, because of the connection with the Leech lattice. Several inequivalent extremal Type II codes over $\mathbb{Z}_4$ have been constructed. The first extremal Type II codes over $\mathbb{Z}_{2k}$ are constructed here for $k = 3$ and 4.

Lifted Golay codes over $\mathbb{Z}_{2^m}$ are given in [4]. We consider a code G_6^{24} of length 24 over $\mathbb{Z}_6$ constructed from the cyclic code with generator polynomial

$$x^{11} + 4x^{10} + x^9 + 2x^8 + 5x^7 + 3x^6 + x^5 + 4x^3 + 3x + 5,$$

by appending 1 to the last coordinate of the generator vectors. The code G_6^{24} is Type II and $G_6^{24} \pmod 2 = \{c \pmod 2 \mid c \in G_6^{24}\}$ is the binary Golay code. G_6^{24} is constructed from binary and ternary cyclic codes.

The swe of the above Type II code G_6^{24} is

$$
\begin{aligned}
&swe_{G_6^{24}}(a,b,c,d) = \\
&d^{24} + 48c^{24} + 36432b^2c^{16}d^6 + 97152b^3c^{12}d^9 \\
&+36432b^4c^8d^{12} + 510048b^5c^{16}d^3 + 3497472b^6c^{12}d^6 + 1603008b^7c^8d^9 \\
&+36432b^8c^{16} + 4048b^9d^{15} + 6800640b^9c^{12}d^3 + 8962272b^{10}c^8d^6 \\
&+61824b^{12}d^{12} + 123648b^{12}c^{12} + 5100480b^{13}c^8d^3 + 242880b^{15}d^9 \\
&+36432b^{16}c^8 + 198352b^{18}d^6 + 24288b^{21}d^3 + 48b^{24} \\
&+13248abc^{11}d^{11} + 971520ab^3c^{15}d^5 + 2914560ab^4c^{11}d^8 + 582912ab^5c^7d^{11} \\
&+4080384ab^6c^{15}d^2 + 36140544ab^7c^{11}d^5 + 14572800ab^8c^7d^8 + 24482304ab^{10}c^{11}d^2 \\
&+39055104ab^{11}c^7d^5 + 8743680ab^{14}c^7d^2 + 145728a^2bc^{14}d^7 + 218592a^2b^2c^{10}d^{10} \\
&+48576a^2b^3c^6d^{13} + 9472320a^2b^4c^{14}d^4 + 24773760a^2b^5c^{10}d^7 + 3934656a^2b^6c^6d^{10} \\
&+8743680a^2b^7c^{14}d + 123868800a^2b^8c^{10}d^4 + 45029952a^2b^9c^6d^7 + 24482304a^2b^{11}c^{10}d \\
&+57076800a^2b^{12}c^6d^4 + 4080384a^2b^{15}c^6d + 4048a^3c^9d^{12} + 24288a^3c^{21} \\
&+2185920a^3b^2c^{13}d^6 + 2963136a^3b^3c^9d^9 + 218592a^3b^4c^5d^{12} + 34974720a^3b^5c^{13}d^3 \\
&+97637760a^3b^6c^9d^6 + 11075328a^3b^7c^5d^9 + 5100480a^3b^8c^{13} + 183941120a^3b^9c^9d^3 \\
&+63391680a^3b^{10}c^5d^6 + 6800640a^3b^{12}c^9 + 34974720a^3b^{13}c^5d^3 + 510048a^3b^{16}c^5 \\
&+36432a^4c^{12}d^8 + 36432a^4bc^8d^{11} + 14427072a^4b^3c^{12}d^5 + 14281344a^4b^4c^8d^8 \\
&+728640a^4b^5c^4d^{11} + 57076800a^4b^6c^{12}d^2 + 187406208a^4b^7c^8d^5 + 15665760a^4b^8c^4d^8 \\
&+123868800a^4b^{10}c^8d^2 + 42989760a^4b^{11}c^4d^5 + 9472320a^4b^{14}c^4d^2 + 582912a^5bc^{11}d^7 \\
&+437184a^5b^2c^7d^{10} + 42989760a^5b^4c^{11}d^4 + 37014912a^5b^5c^7d^7 + 1068672a^5b^6c^3d^{10} \\
&+39055104a^5b^7c^{11}d + 187406208a^5b^8c^7d^4 + 11075328a^5b^9c^3d^7 + 36140544a^5b^{11}c^7d \\
&+14427072a^5b^{12}c^3d^4 + 971520a^5b^{15}c^3d + 198352a^6c^{18} + 3934656a^6b^2c^{10}d^6 \\
&+1603008a^6b^3c^6d^9 + 63391680a^6b^5c^{10}d^3 + 50276160a^6b^6c^6d^6 + 728640a^6b^7c^2d^9 \\
&+8962272a^6b^8c^{10} + 97637760a^6b^9c^6d^3 + 3934656a^6b^{10}c^2d^6 + 3497472a^6b^{12}c^6 \\
&+2185920a^6b^{13}c^2d^3 + 36432a^6b^{16}c^2 + 24288a^7c^9d^8 + 11075328a^7b^3c^9d^5 \\
&+3133152a^7b^4c^5d^8 + 45029952a^7b^6c^9d^2 + 37014912a^7b^7c^5d^5 + 255024a^7b^8cd^8 \\
&+24773760a^7b^{10}c^5d^2 + 582912a^7b^{11}cd^5 + 145728a^7b^{14}cd^2 + 759a^8d^{16} \\
&+255024a^8bc^8d^7 + 15665760a^8b^4c^8d^4 + 3133152a^8b^5c^4d^7 + 14572800a^8b^7c^8d \\
&+14281344a^8b^8c^4d^4 + 24288a^8b^9d^7 + 2914560a^8b^{11}c^4d + 36432a^8b^{12}d^4 \\
&+242880a^9c^{15} + 728640a^9b^2c^7d^6 + 11075328a^9b^5c^7d^3 + 1603008a^9b^6c^3d^6 \\
&+1603008a^9b^8c^7 + 2963136a^9b^9c^3d^3 + 97152a^9b^{12}c^3 + 1068672a^{10}b^3c^6d^5 \\
&+3934656a^{10}b^6c^6d^2 + 437184a^{10}b^7c^2d^5 + 218592a^{10}b^{10}c^2d^2 + 728640a^{11}b^4c^5d^4 \\
&+582912a^{11}b^7c^5d + 36432a^{11}b^8cd^4 + 13248a^{11}b^{11}cd + 2576a^{12}d^{12} \\
&+61824a^{12}c^{12} + 218592a^{12}b^5c^4d^3 + 36432a^{12}b^8c^4 + 4048a^{12}b^9d^3 \\
&+48576a^{13}b^6c^3d^2 + 4048a^{15}c^9 + 759a^{16}d^8 + a^{24}.
\end{aligned}
$$

Thus G_6^{24} is an extremal Type II code of length 24 over $\mathbb{Z}_6$. Applying Theorem 3.1 to G_6^{24}, the Leech lattice is constructed.

Recently some new 5-designs have been constructed from the lifted Golay code over $\mathbb{Z}_4$ (cf. [15]). In addition, any extremal Type II code of length 24 with the same symmetrized weight enumerator as the lifted Golay code contains 5-designs (cf. [1]). $G_6^{24} \pmod 2$ is the binary Golay code and $G_6^{24} \pmod 3$ is an extremal ternary self-dual code. Thus the four sets of the codewords corresponding to $759a^{16}d^8, 4048a^{15}c^9, 2576a^{12}d^{12}$ and $61824a^{12}c^{12}$ form 5-designs. However we have verified by computer that sets of the codewords corresponding to $218592a^{12}b^5c^4d^3$, $36432a^{12}b^8c^4$, $4048a^{12}b^9d^3$ and $48576a^{13}b^6c^3d^2$ do not form 5-designs.

Now we investigate Type II codes of length 24 over $\mathbb{Z}_8$. The lifted Golay codes of length 24 over $\mathbb{Z}_{2^m}$ were constructed from the binary Golay code by the Hensel lifting (cf. [4]). The Golay codes are Type II codes, however the Golay code over $\mathbb{Z}_8$ is not extremal (cf. [9]). In addition, it was shown in [9] that there is no extremal double circulant Type II code over $\mathbb{Z}_8$ of length 24. Thus we consider Type II codes of another type. For $k \geq 2$, a generator matrix of an extremal Type II code over $\mathbb{Z}_{2k}$ give a generator matrix of the Leech lattice. Thus it is natural to investigate generator matrices of the Leech lattice. A generator matrix of the Leech lattice is given in Figure 4.12 of [7]. The generator matrix gives a 23 by 24 matrix $M_{24,8}$ over $\mathbb{Z}_8$. It is easy to see that the matrix $M_{24,8}$ generates a Type II code of length 24. In addition, since the following matrix

$$\frac{1}{\sqrt{8}} \begin{pmatrix} & M_{24,8} & \\ 8 \;\; 0 & \cdots & 0 \end{pmatrix},$$

generates the Leech lattice, the matrix $M_{24,8}$ must generate an extremal Type II code of rank $\{1^1, 2^{11}, 4^{11}\}$. It seems that this code is the first extremal Type II code of length 24 over $\mathbb{Z}_8$.

Proposition 4.1 *There exist extremal Type II codes of length* 24 *over* $\mathbb{Z}_{2k}$ *for* $k \leq 4$.

This gives the following question.

Question. Is there an extremal Type II code of length 24 over $\mathbb{Z}_{2k}$ for $k \geq 5$?

4.2 Methods to Construct Self-Dual Codes

Here we present methods to construct self-dual codes over $\mathbb{Z}_{2k}$.

Proposition 4.2 *Let* $(\ I\ ,\ A\ )$ *be a generator matrix of a Type II code* C *with rank* $\{1^{4n}\}$ *over* $\mathbb{Z}_{2k}$ *of length* $8n$ *containing the all-one vector* $\boldsymbol{j}$ *where* a_i *is the i-th row of* A. *Let* Γ *be a set consisting of* α *columns of* A *where* $0 \leq \alpha \leq 4n$. *Assume that* $|\Gamma|$ *or* k *is even. Let*

$$
M_{24,8} = \left(\begin{array}{cc}
4 & 40000000000000000000000 \\
4 & 04000000000000000000000 \\
4 & 00400000000000000000000 \\
4 & 00040000000000000000000 \\
4 & 00004000000000000000000 \\
4 & 00000400000000000000000 \\
2 & 22222220000000000000000 \\
4 & 00000004000000000000000 \\
4 & 00000000400000000000000 \\
4 & 00000000040000000000000 \\
2 & 22200002222000000000000 \\
4 & 00000000000400000000000 \\
2 & 20022002200220000000000 \\
2 & 02020202020202000000000 \\
2 & 00220022002200200000000 \\
4 & 00000000000000004000000 \\
2 & 02020022200000022000000 \\
2 & 00222002020000020200000 \\
2 & 20020202002000020020000 \\
0 & 22220002000200020002000 \\
0 & 00000002200220022002200 \\
0 & 00000002020202020202020 \\
-3 & 11111111111111111111111
\end{array} \right).
$$

$\boldsymbol{t} = (t_1, \ldots, t_{4n})$ be a $(1,0)$-vector where $t_i = 1$ if $i \in \Gamma$ and $t_i = 0$ otherwise. Let A_Γ be a matrix which has the i-th row

$$
a_i' = \begin{cases} a_i + k\boldsymbol{t}, & \text{if } \|a_i + k\boldsymbol{t}\| \equiv -1 \pmod{4k}, \\ a_i + k\boldsymbol{t} + k\boldsymbol{j}, & \text{otherwise}, \end{cases}
$$

where $\|x\|$ denotes the Euclidean weight of x and $\boldsymbol{j}$ is the all-one's vector. Then the matrix $\boldsymbol{G = (I \ , \ A_\Gamma)}$ generates a Type II code $\boldsymbol{C_\Gamma}$.

Proof. We have $\|a_i + k\boldsymbol{t}\| \equiv \|a_i\| \pmod{2k}$. Moreover, if $\|a_i + k\boldsymbol{t}\| \equiv 2k-1 \pmod{4k}$, then $\|a_i + k\boldsymbol{t} + k\boldsymbol{j}\| \equiv -1 \pmod{4k}$. Thus a row of G is orthogonal to itself and the Euclidean weights of all the rows of G are divisible by $4k$. In addition, the i-th row a_i' of A_Γ can be written as

$$
a_i' = a_i + k\boldsymbol{t} + k\boldsymbol{j}\left(\frac{\|a_i + k\boldsymbol{t}\| + 1}{2k}\right).
$$

Since C contains the all-one vector, $a_i \cdot \boldsymbol{j} \equiv -1 \pmod{2k}$. Thus we have

$$
\begin{aligned}
< a_i', a_j' > &= < (a_i + k\boldsymbol{t} + k\boldsymbol{j}(\frac{\|a_i + k\boldsymbol{t}\| + 1}{2k})), (a_j + k\boldsymbol{t} + k\boldsymbol{j}(\frac{\|a_j + k\boldsymbol{t}\| + 1}{2k})) > \\
&= < a_i, a_j > + < ka_i, \boldsymbol{t} > + < ka_j, \boldsymbol{t} > + < (\frac{\|a_i + k\boldsymbol{t}\| + 1}{2k}), (\frac{\|a_j + k\boldsymbol{t}\| + 1}{2k}) > \\
&= < a_i, a_j > .
\end{aligned}
$$

Therefore the code C_Γ is self-dual.

The Euclidean weight of a row of G is divisible by $4k$ and C is self-dual. Thus it follows from Lemma 2.2 that the Euclidean weight of every codeword of the code is divisible by $4k$.

□

Starting one generator matrix, one can construct a number of Type II codes which might be inequivalent codes.

Corollary 4.3 *Let the assumptions and notations be the same as ones of Proposition 4.2. Let B_Γ be a matrix which has the i-th row*

$$b'_i = \begin{cases} a_i + k\boldsymbol{t}, & \text{if } \|a_i + k\boldsymbol{t}\| \equiv 2k-1 \pmod{4k}, \\ a_i + k\boldsymbol{t} + k\boldsymbol{j}, & \text{otherwise.} \end{cases}$$

Then the matrix $G' = (\ I\ ,\ B_\Gamma\ )$ generates a Type I code C'_Γ.

Remark. We gave methods to construct Type I and Type II codes from certain Type II codes. Similarly one can easily get similar methods to construct Type I and Type II codes from Type I codes of length $8n$.

As an example, we construct an extremal Type I code over $\mathbb{Z}_4$ of length 24. An extremal Type II code D_{24} over $\mathbb{Z}_4$ with generator matrix of the form

$$\left(\begin{array}{c|c|ccc} & 2 & 3 & \cdots & 3 \\ & 1 & & & \\ I & \vdots & & R & \\ & 1 & & & \end{array} \right),$$

where R is a 24 by 24 circulant matrix with first row (21311133313) is given in [5]. By Corollary 4.3, Type I codes are constructed from D_{24}. When $\Gamma = \{1\}$, it is easy to see that its generator matrix is

$$G = \left(\begin{array}{c|c|ccc} & 0 & 3 & \cdots & 3 \\ & 1 & & & \\ I & \vdots & & R+2J & \\ & 1 & & & \end{array} \right),$$

and the minimum Euclidean weight of this code is 12, thus this is an extremal Type I code over $\mathbb{Z}_4$ of length 24. By Theorem 3.1, this code yields the 24-dimensional unique odd unimodular lattice with minimum norm 3 which is called the odd Leech lattice. Other extremal Type I codes of length 24 are constructed in [13].

5 Weight Enumerators, MacWilliams Identities and Invariants

In this section, we introduce several types of weight enumerators of codes over $\mathbb{Z}_{2k}$. For these weight enumerators, we establish the MacWilliams identities and study invariants. From now on R denotes the ring $\mathbb{Z}_{2k}$.

5.1 Weight Enumerators and MacWilliams Identities

First let us fix the notations. We denote the primitive m-th root $e^{2\pi i/m}$ of unity by η_m. $A\,[B] := {}^tABA$ for matrices A and B, where ${}^tA$ denotes the transpose of A.

Definition 1 (Complete Weight Enumerators in Genus g) *For a code C over R, we define the complete weight enumerator in genus g by*

$$\mathfrak{C}_{C,g}(z_a \text{with } a \in R^g) = \sum_{c_1,\dots,c_g \in C} \prod_{a \in R^g} z_a^{n_a(c_1,\dots,c_g)},$$

where $n_a(c_1,\dots,c_g)$ denotes the number of i satisfying $a = {}^t(c_{1i},\dots,c_{gi})$.

Remark. (1) For the case $g = 1$, these weight enumerators are the same as ordinary complete weight enumerators defined in Section 2.
(2) For the case $k = 1$ these weight enumerators were introduced in [12] and [23].

We define a relation $\sim$ in R^g by

$$a \sim b \Longleftrightarrow a = b \text{ or } a = -b$$

where $a, b \in R^g$. Then the relation $\sim$ becomes an equivalence relation in R^g and we denote the natural projection using the conventions $\overline{a}$. Note that $wt_E(a) = wt_E(-a)$ and $wt_L(a) = wt_L(-a)$.

Definition 2 (Symmetrized Weight Enumerators in Genus g) *For a code C over R, we define the symmetrized weight enumerator in genus g by*

$$\mathfrak{S}_{C,g}(z_{\overline{a}} \text{with } \overline{a} \in \overline{R^g}) = \sum_{c_1,\dots,c_g \in C_g} \prod_{\overline{a} \in \overline{R^g}} z_{\overline{a}}^{n_{\overline{a}}(c_1,\dots,c_g)},$$

where $n_{\overline{a}}(c_1,\dots,c_g)$ denotes the number of i satisfying $\overline{a} = \overline{{}^t(c_{1i},\dots,c_{gi})}$.

Remark. For the case $g = 1$, these weight enumerators are the same as ordinary symmetrized weight enumerators defined in Section 2.

From now on, we often write complete and symmetrized weight enumerators in genus g by $\mathfrak{C}_{C,g}(z_a)$, $\mathfrak{S}_{C,g}(z_{\overline{a}})$, respectively, for simplicity.

We have the *MacWilliams identity* for the complete weight enumerators. Here we consider that an n by n matrix M acts on the polynomial ring $\mathbb{C}[x_1, x_2, \dots, x_n]$ naturally, that is,

$$M \cdot f(x_1, x_2, \dots, x_n) = f(\sum_{1 \le j \le n} a_{1j}x_j, \dots, \sum_{1 \le j \le n} a_{nj}x_j),$$

where $f \in \mathbb{C}[x_1, x_2, \dots, x_n]$ and $A = (a_{ij})$.

Theorem 5.1 (MacWilliams Identity) *For a code C over R, we have*

$$\mathfrak{C}_{C^\perp,g}(z_a) = \frac{1}{|C|^g} T \cdot \mathfrak{C}_{C,\,g}(z_a),$$

where $T = \left(\eta_{2k}^{\langle a,b\rangle}\right)_{a,b\in R^g}$.

Similarly, we have the *MacWilliams identity* for the symmetrized weight enumerators.

Corollary 5.2 (MacWilliams Identity) *For a code C, we have*

$$\mathfrak{S}_{C^\perp,g}(z_{\overline{a}}) = \frac{1}{|C|^g} \overline{T} \cdot \mathfrak{S}_{C,g}(z_{\overline{a}}),$$

where $\overline{T} = \left(t(\overline{a},\overline{b})\right)_{\overline{a},\overline{b}\in\overline{R^g}}$, *and* $t(\overline{a},\overline{b}) = \sum_{d\in R^g \text{ with } \overline{d}=\overline{b}} \eta_{2k}^{\langle a,d\rangle}$.

5.2 Invariant Rings

In this subsection, we study the invariance properties of complete and symmetrized weight enumerators.

We define a subgroup $G^8_{g,k}$ of $GL((2k)^g, \mathbb{C})$ as

$$G^8_{g,k} = \langle T_g, D_S, \eta_8 |\ S \text{ runs over all integral symmetric matrices}\rangle,$$

where

$$T_g = \left(\frac{\eta_8}{\sqrt{2k}}\right)^g T, D_S = \text{ diag } (\eta_{4k}^{S[a]} \text{ with } a \in R^g).$$

Theorem 5.3 *For any Type II code C over R, the complete weight enumerator in genus g is invariant under the action of the group $G^8_{g,k}$.*

Proof. We have only to check three types of generators, T_g, D_S, and η_8. The invariance property of T_g, η_8 comes from Corollary 3.3 and Theorem 5.1. We shall show that $D_S \cdot \mathfrak{C}_{C,g}(z_a) = \mathfrak{C}_{C,g}(z_a)$. We have

$$\begin{aligned} D_S \cdot \mathfrak{C}_{C,\,g}(z_a) &= \sum_{c_1,\ldots,c_g\in C} \prod_{a\in R} (\eta_{4k}^{S[a]} z_a)^{n_a(c_1,\ldots,c_g)} \\ &= \sum_{c_1,\ldots,c_g\in C} \prod_{a\in R} \eta_{4k}^{S[a]\cdot n_a(c_1,\ldots,c_g)} z_a^{n_a(c_1,\ldots,c_g)}. \end{aligned}$$

In order to prove the theorem, we have to show $\sum_{a\in R} S[a]\cdot n_a(c_1,\ldots,c_g) \equiv 0 \pmod{4k}$.

$$\begin{aligned} \sum_{a\in R} S[a]\cdot n_a(c_1,\ldots,c_g) &= \sum_{1\le i\le n} S[{}^t(c_{1i},\ldots,c_{gi})] \\ &= \sum_{1\le i\le n} \left\{ \sum_{1\le k\le g} s_{kk}(c_{ki})^2 + 2 \sum_{1\le l<m\le g} s_{lm} c_{li}\, c_{mi} \right\} \\ &= \sum_{1\le k\le g} s_{kk} \sum_{1\le i\le n} (c_{ki})^2 + 2 \sum_{1\le l<m\le g} s_{lm} \sum_{1\le i\le n} c_{li}\, c_{mi}. \end{aligned}$$

For any element c_k, we have $wt_E(c_k) = \sum_{1\le i\le n}(c_{ki})^2 \equiv 0 \pmod{4k}$. And $\sum_{1\le i\le n} c_{li}c_{ki} \equiv 0 \pmod{2k}$ follows from the calculation $d_E(c_l, c_k) \equiv 0 \pmod{4k}$. Therefore it turns out that $\sum_{a\in R} S[a]\cdot n_a(c_1,\ldots,c_g) \equiv 0 \pmod{4k}$. This completes the proof of the theorem. □

Remark. (1) $G_{g,k}$ is (up to ± 1) the homomorphic image of the modular group Γ_g under the theta representation of index k (cf. [24]).
(2) Theorem 5.3 says that the ring generated by complete weight enumerators for Type II codes is contained the invariant ring of the group $G^8_{g,k}$. For $k=1$, two rings coincide (cf. Theorem 3.6 in [23]).

We now define a subgroup $H^8_{g,k}$ of $GL(2^{g-1}(k^g+1),\mathbb{C})$ as

$$H^8_{g,k} = \langle \overline{T_g}, \overline{D_S}, \eta_8 |\ S \text{ runs over all integral symmetric matrices } \rangle,$$

where

$$\overline{T_g} = \left(\frac{\eta_8}{\sqrt{2k}}\right)^g \overline{T} \text{ and } \overline{D_S} = \text{ diag } (\eta_{4k}^{S[a]} \text{ with } \overline{a} \in \varphi(R^g)).$$

Similarly to complete weight enumerators, we have the following MacWilliams identity for symmetrized weight enumerators in genus g.

Corollary 5.4 *For any Type II code C over R, the symmetrized weight enumerator in genus g is invariant under the action of the group $H^8_{g,k}$.*

In concluding this subsection, we would like to emphasize that the groups $G_{g,k}$, as well the groups $H_{g,k}$, $G^8_{g,k}$ and $H^8_{g,k}$, are all finite groups. This is explained as follows. Here we assume that the reader is familiar with some of the basic concepts of theta functions, such as given in Runge [24].

The group $H_{g,k} =< \overline{T_g}, \overline{D_S} \mid S$ runs over all integral symmetric matrices $>$ acts linearly on the space spanned by the theta constants $f_a^{(k)}$ of index k, where

$$f_a^{(k)}(\tau) = \sum_{x\in\mathbb{Z}^g} \exp 2\pi i (k\tau[x+\frac{a}{2k}])$$

Note that here $k\in\mathbb{N}, a\in(\mathbb{Z}_{2k})^g$.

It is known that the group $H_{g,k}/(\pm 1)$ is a homomorphic image of the Siegel modular group $\Gamma_g = Sp(2g,\mathbb{Z})$ under the theta representation of index k:

$$\rho_{theta,k} : \Gamma_g \longrightarrow Aut(\mathcal{TH}^{(k)}_{g,(2)})$$

in the notation of [24]. The kernel of this representation is completely described in Runge [24, Theorem 2.4]. In particular, this kernel contains the subgroup $\Gamma_g(4k)$. Since $\Gamma_g/\Gamma_g(4k) \cong Sp(2g,\mathbb{Z}_{4k})$ is a finite group, the finiteness of the group $H_{g,k}$ follows immediately.

Similarly, the group $G_{g,k} =< T_g, D_S \mid S$ as above $>$ acts linearly on the space spanned by the theta functions $f_a^{(k)}(\tau, z)$ of index k, where

$$f_a^{(k)}(\tau, z) = \sum_{x \in \mathbb{Z}^g} \exp 2\pi i (k\tau[x + \frac{a}{2k}] + < x + \frac{a}{2k}, 2kz >).$$

Again, $G_{g,k}/(\pm 1)$ is a homomorphic image of $\Gamma_g = Sp(2g, \mathbb{Z})$ under the theta representation:

$$\rho_{theta,k} : \Gamma_g \longrightarrow Aut(\mathcal{THET}_{g,(2)}^{(\leq k)})$$

in the notation of [24]. From the relation

$$\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \cdot f_a^{(k)}(\tau, z) = \exp 2\pi i (\frac{S[a]}{4k}) \cdot f_a^{(k)}(\tau, z),$$

it is again proved that $\Gamma(4k)$ is in the kernel of the theta representation, see e.g., Runge [24] or Kac [17, Theorem 13.5 (p. 169)]. Since the group $\Gamma_g/\Gamma_g(4k)$ is finite and since $|G_{g,k}| \leq 2 \cdot |\Gamma_g/\Gamma_g(4k)|$, we have the finiteness of the group $G_{g,k}$. The finiteness of the groups $G_{g,k}^8$ and $H_{g,k}^8$ are immediately obtained as $|G_{g,k}^8| \leq 8 \cdot |G_{g,k}|$ and $|H_{g,k}^8| \leq 8 \cdot |H_{g,k}|$.

Although we will not discuss the details here, it is possible to determine the orders and the structures of the groups $G_{g,k}$, $H_{g,k}$, $G_{g,k}^8$ and $H_{g,k}^8$ more explicitly, by using the known explicit determinations of the kernels of the theta representations $\rho_{theta,k} : \Gamma_g \longrightarrow Aut(\mathcal{TH}_{g,(2)}^{(k)})$ given in Runge [24].

We give in Table 1 the orders of the groups $G_{g,k}$, $H_{g,k}$, $G_{g,k}^8$ and $H_{g,k}^8$ for $g = 1$ and $k \leq 8$. It can be shown, for example, that

$$|G_{1,2^m}^8| = 192 \cdot 2^{m-1}.$$

Table 1: Orders of the Groups $G_{g,k}$, $H_{g,k}$, $G_{g,k}^8$ and $H_{g,k}^8$

k	1	2	3	4	5	6	7	8
$\|G_{1,k}\|$	96	384	2304	3072	11520	9216	32256	24576
$\|H_{1,k}\|$	96	384	1152	3072	5760	9216	16128	24576
$\|G_{1,k}^8\|$	192	1536	4608	12288	23040	368648	64512	98304
$\|H_{1,k}^8\|$	192	768	2304	6144	11520	18432	32256	49152

6 Shadows and Weight Enumerators

We first prove that the complete (resp. symmetrized) weight enumerator of the shadow of a Type I code C over $\mathbb{Z}_{2k}$ is uniquely determined from the complete (resp. symmetrized) weight enumerator of C.

Lemma 6.1 *If C is a Type I code over $\mathbb{Z}_{2k}$ then*

$$\begin{aligned} cwe_{C_0}(x_0, x_1, \ldots, x_{2k-1}) &= \frac{1}{2}(swe_C(x_0, x_1, \ldots, x_{2k-1}) + swe_C(\eta_{4k}^{0^2} x_0, \eta_{4k}^{1^2} x_1, \ldots, \eta_{4k}^{(2k-1)^2} x_{2k-1})) \\ swe_{C_0}(x_0, x_1, \ldots, x_k) &= \frac{1}{2}(swe_C(x_0, x_1, \ldots, x_k) + swe_C(\eta_{4k}^{0^2} x_0, \eta_{4k}^{1^2} x_0, \ldots, \eta_{4k}^{k^2} x_k)). \end{aligned}$$

where η_{4k} denotes the primitive $4k$-th root of unity.

Proof. Let $c = (c_1, c_2, \ldots, c_n)$ be a codeword in C then

$$\prod_{i=1}^{n} (\eta_{4k}^{i^2} x_i)^{n_i(c)} = \prod_{i=1}^{n} (\eta_{4k})^{i^2 n_i(c)} \prod_{i=1}^{n} (x_i)^{n_i(c)} = (\eta_{4k})^{\sum_{i=1}^{n} i^2 n_i(c)} \prod_{i=1}^{n} (x_i)^{n_i(c)}.$$

Since C is self-dual, c has Euclidean weight $\equiv 0 \pmod{2k}$. Since $wt_E(c) \equiv \sum_{i=1}^{n} i^2 n_i(c) \pmod{4k}$,

$$\prod_{i=1}^{n} (\eta_{4k}^{i^2} x_i)^{n_i(c)} = \begin{cases} -\prod_{i=1}^{n} x_i^{n_i(c)} & \text{if } wt_E(c) \equiv 2k \pmod{4k} \\ \prod_{i=1}^{n} x_i^{n_i(c)} & \text{if } wt_E(c) \equiv 0 \pmod{4k}. \end{cases}$$

This proves the lemma. The swe is computed from the cwe. □

Theorem 6.2 *Let C be a Type I code over $\mathbb{Z}_{2k}$ and let S be its shadow. Then the cwe and swe of S is related to the cwe and swe of C by the relation*

$$\begin{aligned} cwe_S(x_0, x_1, \ldots, x_{2k-1}) &= cwe_C(A(x_0, x_1, \ldots, x_{2k-1})) \\ swe_S(x_0, x_1, \ldots, x_k) &= swe_C(B(x_0, x_1, \ldots, x_k)) \end{aligned}$$

where $A = (a_{ij})$ is the $2k$ by $2k$ matrix with $a_{ij} = \frac{1}{\sqrt{2k}} \eta_{4k}^{i^2+2ij}$, and $B = (b_{ij})$ is the $(k+1)$ by $(k+1)$ matrix with $b_{ij} = \sum_{i' \equiv i} a_{i'j}$ where $i' \equiv i$ if $i' = i$ or $i' = -i$.

Proof. We proceed as in [6, p. 1323] by computing first by the MacWilliams identity

$$cwe_{C^\perp}(x_0, x_1, \ldots, x_{2k-1}) = \frac{1}{|C|} cwe_C(M(x_0, x_1, \ldots, x_{2k-1})$$

where $M = (m_{ij})$ is the $2k$ by $2k$ matrix with $m_{ij} = \eta_{2k}^{ij}$, the cwe of $C^\perp$, then the cwe of its $4k$-weight subcode, the cwe of the dual of the latter, and finally the cwe of the shadow by the difference of the cwe of $C_0^\perp$ and the cwe of C. The swe follows similarly. □

Definition 3 (Complete Joint Weight Enumerators) *The complete joint weight enumerator for codes C and K of length n over R is defined as*

$$\mathfrak{J}_{C,K}(X_{\mathbf{a}} \text{ with } \mathbf{a} \in R \times R) = \sum_{(c,k) \in C \times K} \prod_{\mathbf{a} \in R \times R} X_{\mathbf{a}}^{n_{\mathbf{a}}(c,k)}$$

where $n_{\mathbf{a}}(c, k) = |\{j | (c_j, k_j) = \mathbf{a}\}|$, $c = (c_1, \ldots, c_n)$ and $k = (k_1, \ldots, k_n)$. Similarly to complete weight enumerators, we often simply denote the weight enumerators by $\mathfrak{J}_{C,K}(X_{\mathbf{a}})$.

In a similar argument to Theorem 5.1, we have the MacWilliams identity for complete joint weight enumerators.

Theorem 6.3 (MacWilliams Identity) *Let $\tilde{A}$ denote either A or $A^\perp$. Then*

$$\mathfrak{J}_{\tilde{C},\tilde{K}}(X_{\mathbf{a}}) = \frac{1}{|C|^{\delta_{\tilde{C},C^\perp}}|K|^{\delta_{\tilde{K},K^\perp}}}(T^{\delta_{\tilde{C},C^\perp}} \otimes T^{\delta_{\tilde{K},K^\perp}})\mathfrak{J}_{C,K}(X_{\mathbf{a}}),$$

where

$$T = \left(\eta_{2k}^{\langle a,b\rangle}\right)_{a,b\in R} \text{ and } \delta_{\tilde{A},A^\perp} = \begin{cases} 0 & \text{if } \tilde{A} = A, \\ 1 & \text{if } \tilde{A} = A^\perp. \end{cases}$$

Proof. Similar to that of Theorem 5.1. □

We give relationships between a Type I code and its shadow using the weight enumerators.

Given the complete joint weight enumerator for $\mathfrak{J}_{C,C}$ we can find $\mathfrak{J}_{C,C_0}$, $\mathfrak{J}_{C_0,C}$, and $\mathfrak{J}_{C_0,C_0}$.

Proposition 6.4 *Let C be a Type I code over R and let C_0 be the $4k$-weight subcode of C. Then*

$$\begin{aligned}
\mathfrak{J}_{C,C_0}(X_{\mathbf{a}}) &= \frac{1}{2}(\mathfrak{J}_{C,C}(X_{\mathbf{a}}) + \mathfrak{J}_{C,C}(X_{\phi(\mathbf{a})})) \\
\mathfrak{J}_{C_0,C}(X_{\mathbf{a}}) &= \frac{1}{2}(\mathfrak{J}_{C,C}(X_{\mathbf{a}}) + \mathfrak{J}_{C,C}(X_{\psi(\mathbf{a})})) \\
\mathfrak{J}_{C_0,C_0}(X_{\mathbf{a}}) &= \frac{1}{4}(\mathfrak{J}_{C,C}(X_{\mathbf{a}}) + \mathfrak{J}_{C,C}(X_{\phi(\mathbf{a})}) + \mathfrak{J}_{C,C}(X_{\psi(\mathbf{a})}) + \mathfrak{J}_{C,C}(X_{\theta(\mathbf{a})}))
\end{aligned}$$

where $\phi(\mathbf{a}) = \eta_{4k}^{b^2}(a,b)$, $\psi(\mathbf{a}) = \eta_{4k}^{a^2}(a,b)$ and $\theta(\mathbf{a}) = \eta_{4k}^{a^2+b^2}(a,b)$ for $\mathbf{a} = (a,b) \in R \times R$.

Proof. Notice that the substitution $X_{\phi(\mathbf{a})}$ fixes each monomial representing codewords with Euclidean weight divisible by $4k$ and negates each monomial representing codewords whose Euclidean weight $\equiv 2k \pmod{4k}$, which gives the result. The remaining two cases are similar. □

We can apply the MacWilliams identity to find all the joint weight enumerators involving $C, C_0, C_0^\perp$, and S. In particular we have the following:

Proposition 6.5 *Let C be a Type I code over R and let S be its shadow then*

$$\begin{aligned}
\mathfrak{J}_{S,C}(X_{\mathbf{a}}) &= (T \otimes I)\,\mathfrak{J}_{C,C}(X_{\phi(\mathbf{a})}) \\
\mathfrak{J}_{C,S}(X_{\mathbf{a}}) &= (I \otimes T)\,\mathfrak{J}_{C,C}(X_{\psi(\mathbf{a})}) \\
\mathfrak{J}_{S,S}(X_{\mathbf{a}}) &= (T \otimes T)\,\mathfrak{J}_{C,C}((X_{\theta(\mathbf{a})}).
\end{aligned}$$

Proof. We compute $\mathfrak{J}_{C,C_0}$, $\mathfrak{J}_{C_0,C}$ and $\mathfrak{J}_{C_0,C_0}$ by the above theorem, apply the MacWilliams identity and then compute the desired weight enumerators from these weight enumerators. $\square$

Lemma 6.1, Theorem 6.2, Propositions 6.4 and 6.5 determine complete, symmetrized and joint weight enumerators for C_0 and S from ones of C. For the code to exist all of these weight enumerators must have non-negative integral coefficients. Our results seem to be useful for proving the non-existence of a certain Type I code over $\mathbb{Z}_{2k}$. In fact, for the case $k = 1$ the non-existence of some Type I codes with high minimum weight was proved in [6] using their shadows.

7 Construction of Siegel Modular Forms

We first recall the notations of theta functions (for more detail, e.g. see [24]):

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau) := \sum_{x \in \mathbb{Z}^g} \exp 2\pi\sqrt{-1} \left(\frac{1}{2} \tau \left[x + \frac{\alpha}{2} \right] + \langle x + \frac{\alpha}{2}, \frac{\beta}{2} \rangle \right), \ \alpha, \beta \in \mathbb{F}_2^g, \ \tau \in \mathcal{H}_g,$$

where $\mathcal{H}_g$ denotes the Siegel upper half-space $\mathcal{H}_g = \{Z = X + iY \in GL(g, \mathbb{C}) | Z = {}^tZ, Y > 0\}$.

We define for any positive integer k the following theta functions:

$$f_a^{(k)}(\tau) := \theta \begin{bmatrix} a/k \\ 0 \end{bmatrix} (2k\tau).$$

It is well known that the modular group $\Gamma_g = Sp(2g, \mathbb{Z})$ is generated by the elements $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and $D_S = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}$, where S runs over the symmetric g by g matrices. They act on the theta functions as follows:

$$D_S(f_a^{(k)})(\tau) = \exp 2\pi i \left(\frac{S[a]}{4k} \right) f_a^{(k)}(\tau), \ \ \frac{J(f_a^{(k)})(\tau)}{\sqrt{\det(-\tau)}} = \sum_{b \in (\mathbb{Z}_{2k})^g} (T_g)_{a,b} f_b^{(k)}(\tau).$$

Moreover, the theta functions for a lattice L are defined by

$$\theta_{L,g}(\tau) := \sum_{x_1,\ldots,x_g \in L} \prod_{1 \le i,j \le g} q_{ij}^{[x_i, x_j]}$$

where $q_{ij} = \exp \pi\sqrt{-1}\tau_{ij}$.

A Siegel modular form of weight k for $\Gamma_g = Sp(2g, \mathbb{Z})$ is a holomorphic function f on the Siegel upper half-space such that for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$, we have

$$f((A\tau + B)(C\tau + D)^{-1}) = \det(C\tau + D)^k f(\tau).$$

We need more conditions for the case $g = 1$.

Theorem 7.1 *Let C be a Type II code of length n over R and let $\Lambda(C)$ be the even unimodular lattice constructed from C by Theorem 3.1. Then*

$$\mathfrak{C}_{C,g}(f_a^{(k)}(\tau)) = \mathfrak{S}_{C,g}(f_{\overline{a}}^{(k)}(\tau)) = \theta_{\Lambda(C),g}(\tau)$$

and these functions give Siegel modular forms of weight $n/2$ for Γ_g.

8 Molien Series for Small Cases

The weight enumerator of a self-dual code belongs to the ring of polynomials fixed by the group of substitutions. In this section, we give the Molien series for the invariant rings of the groups of small k and g.

First, let us recall the general invariant theory of finite groups. Let G be a finite subgroup of $GL(n, \mathbb{C})$. Then G acts on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ ($\mathbb{C}[x_k]$ for short) naturally, i.e.,

$$A \cdot f(x_1, \ldots, x_n) = f(\sum_{1\le j\le n} A_{1j}x_j, \ldots, \sum_{1\le j\le n} A_{nj}x_j),$$

where $f \in \mathbb{C}[x_k]$ and $A = (A_{ij})_{1\le i,j\le n}$. There exists a *homogeneous system of parameters* $\{\theta_1, \ldots, \theta_n\}$ such that the invariant ring $\mathbb{C}[x_k]^G$ is finitely generated free $\mathbb{C}[\theta_1, \ldots, \theta_n]$-module. The invariant ring has the *Hironaka decomposition*

$$\mathbb{C}[x_k]^G = \oplus_{1\le m\le s} g_m \mathbb{C}[\theta_1, \ldots, \theta_n],\ g_1 = 1.$$

The invariant ring is an graded ring and the dimension formula is defined by

$$\Phi_G(t) = \sum_{d\ge 1} \dim \mathbb{C}[x_k]_d^G t^d,$$

where $\mathbb{C}[x_k]_d^G$ is the d-th homogeneous part of $\mathbb{C}[x_k]^G$. The dimension formula for the *Hironaka decomposition* given in the above form is

$$\Phi_G(t) = \frac{1 + t^{\deg(g_2)} + \cdots + t^{\deg(g_s)}}{(1 - t^{\deg(\theta_1)}) \cdots (1 - t^{\deg(\theta_n)})}.$$

In general, the converse is not true. It is known that we have the identity

$$\Phi_G(t) = \sum_{A\in G} \frac{1}{\det(1 - tA)},$$

This was shown by Molien and is called *Molien series.*

We recall the notations:

$$\begin{aligned} R &:= \mathbb{Z}_{2k} \\ G_{g,k} &:= \left\langle \left(\frac{\eta_8}{\sqrt{2k}}\right)^g \left(\eta_{2k}^{\langle a,b\rangle}\right)_{a,b\in R^g},\ \text{diag}\ (\eta_{4k}^{S[a]} \text{ with } a \in R^g)\right\rangle \end{aligned}$$

$$
\begin{aligned}
G_{g,k}^8 &:= \langle G_{g,k}, \eta_8 \rangle \\
H_{g,k} &:= \left\langle \left(\frac{\eta_8}{\sqrt{2k}}\right)^g \left(t(\overline{a},\overline{b})\right)_{\overline{a},\overline{b}\in\overline{R^g}},\ \mathrm{diag}\ (\eta_{4k}^{S[a]} \text{ with } \overline{a}\in\overline{R^g}) \right\rangle, \\
H_{g,k}^8 &:= \langle H_{g,k}, \eta_8 \rangle,
\end{aligned}
$$

where $t(\overline{a},\overline{b}) = \sum_{d\in R^g \text{ with } \overline{d}=\overline{b}} \eta_{2k}^{\langle a,d\rangle}$.

In the following, we give the Molien series in the form

$$
\begin{aligned}
\Phi_G(t) &= \text{the expansion} \\
&= \text{the Hironaka decomposition} \\
&= \text{the Hironaka decomposition with factored numerators.}
\end{aligned}
$$

If the numerator is irreducible, we omit the third line for each case.

$|G_{1,1}| = 96$ and

$$
\begin{aligned}
\Phi_{G_{1,1}}(t) &= 1 + t^8 + t^{12} + t^{16} + t^{20} + 2t^{24} + t^{28} + 2t^{32} + \cdots \\
&= 1/(1-t^8)(1-t^{12})
\end{aligned}
$$

$|G_{1,2}| = 384$ and

$$
\begin{aligned}
\Phi_{G_{1,2}}(t) &= 1 + 4t^8 + 2t^{10} + 3t^{12} + 2t^{14} + 11t^{16} + 7t^{18} + \\
&\quad 11t^{20} + 9t^{22} + 25t^{24} + 18t^{26} + 27t^{28} + 23t^{30} + 48t^{32} + \cdots \\
&= (1 + t^8 + 2t^{10} + 2t^{12} + 2t^{14} + 2t^{16} + t^{18} + t^{20} + t^{22} + \\
&\quad t^{26} + t^{28} + t^{30})/(1-t^8)^3(1-t^{12}) \\
&= \left(1+t^2\right)\left(1+t^4\right) \\
&\quad \left(1 - t^2 + 2t^8 + 2t^{16} - t^{18} + t^{24}\right)/(1-t^8)^3(1-t^{12})
\end{aligned}
$$

$|G_{1,3}| = 2304$ and

$$
\begin{aligned}
\Phi_{G_{1,3}}(t) &= 1 + t^8 + 15t^{12} + 37t^{16} + 78t^{20} + 229t^{24} + 419t^{28} + 721t^{32} + \cdots \\
&= (1 + 12t^{12} + 36t^{16} + 63t^{20} + 148t^{24} + 233t^{28} + 303t^{32} + 366t^{36} + \\
&\quad 444t^{40} + 460t^{44} + 427t^{48} + 338t^{52} + 272t^{56} + 174t^{60} + 96t^{64} + 53t^{68} \\
&\quad + 24t^{72} + 5t^{76} + t^{80})/(1-t^8)(1-t^{12})^3(1-t^{24})^2 \\
&= \left(1 - t + t^2\right)\left(1 + t + t^2\right)\left(1 - t^2 + t^4\right)(1 - t^4 + 13t^{12} + 23t^{16} + 27t^{20} + \\
&\quad 98t^{24} + 108t^{28} + 97t^{32} + 161t^{36} + 186t^{40} + 113t^{44} + 128t^{48} + 97t^{52} + \\
&\quad 47t^{56} + 30t^{60} + 19t^{64} + 4t^{68} + t^{72})/(1-t^8)(1-t^{12})^3(1-t^{24})^2
\end{aligned}
$$

$|G_{1,1}^8| = 192$ and

$$
\begin{aligned}
\Phi_{G_{1,1}^8}(t) &= 1 + t^8 + t^{16} + 2t^{24} + 2t^{32} + \cdots \\
&= 1/(1-t^8)(1-t^{24})
\end{aligned}
$$

$|G^8_{1,2}| = 1536$ and

$$\begin{aligned}\Phi_{G^8_{1,2}}(t) &= 1 + 4t^8 + 11t^{16} + 25t^{24} + 48t^{32} + \cdots \\ &= (1 + t^8 + 2t^{16} + 2t^{24} + t^{32} + t^{40})/(1 - t^8)^3(1 - t^{24}) \\ &= (1 + t^8)(1 + t^{16})^2/(1 - t^8)^3(1 - t^{24})\end{aligned}$$

$|G^8_{1,3}| = 4608$ and

$$\begin{aligned}\Phi_{G^8_{1,3}}(t) &= 1 + t^8 + 37t^{16} + 229t^{24} + 721t^{32} + \cdots \\ &= (1 + 35t^{16} + 188t^{24} + 456t^{32} + 1099t^{40} + 1677t^{48} + \\ &\quad 1829t^{56} + 1793t^{64} + 1246t^{72} + 590t^{80} + 241t^{88} + 56t^{96} + \\ &\quad 5t^{104})/(1 - t^8)(1 - t^{16})(1 - t^{24})^4 \\ &= (1 + t^8)(1 - t^8 + 36t^{16} + 152t^{24} + 304t^{32} + 795t^{40} + \\ &\quad 882t^{48} + 947t^{56} + 846t^{64} + 400t^{72} + 190t^{80} + 51t^{88} + \\ &\quad 5t^{96})/(1 - t^8)(1 - t^{16})(1 - t^{24})^4\end{aligned}$$

$|H_{1,2}| = 384$ and

$$\begin{aligned}\Phi_{H_{1,2}}(t) &= 1 + 2t^8 + t^{12} + 4t^{16} + 2t^{20} + 7t^{24} + 4t^{28} + 10t^{32} + \cdots \\ &= (1 + t^{16})/(1 - t^8)^2(1 - t^{12})\end{aligned}$$

$|H_{1,3}| = 1152$ and

$$\begin{aligned}\Phi_{H_{1,3}}(t) &= 1 + t^8 + 3t^{12} + 4t^{16} + 5t^{20} + 15t^{24} + \\ &\quad 14t^{28} + 24t^{32} + \cdots \\ &= (1 + 2t^{12} + 3t^{16} + 2t^{20} + 6t^{24} + 6t^{28} + 7t^{32} + 6t^{36} + \\ &\quad 5t^{40} + 6t^{44} + t^{48} + 2t^{52} + t^{56})/(1 - t^8)(1 - t^{12})(1 - t^{24})^2 \\ &= (1 - t + t^2)(1 + t + t^2)(1 + t^4)(1 - t^2 + t^4)(1 - t^4 + t^8) \\ &\quad (1 - t^4 + 2t^{12} + t^{16} - t^{20} + 5t^{24} - t^{28} + t^{32} + t^{36}) \\ &\quad /(1 - t^8)(1 - t^{12})(1 - t^{24})^2\end{aligned}$$

$|H^8_{1,2}| = 768$ and

$$\begin{aligned}\Phi_{H^8_{1,2}}(t) &= 1 + 2t^8 + 4t^{16} + 7t^{24} + 10t^{32} + \cdots \\ &= (1 + t^{16})/(1 - t^8)^2(1 - t^{24})\end{aligned}$$

$|H^8_{1,3}| = 2304$ and

$$\begin{aligned}\Phi_{H^8_{1,3}}(t) &= 1 + t^8 + 4t^{16} + 15t^{24} + 24t^{32} + \cdots \\ &= (1 + 2t^{16} + 9t^{24} + 6t^{32} + 5t^{40} + 7t^{48} + 2t^{56}) \\ &\quad /(1 - t^8)(1 - t^{16})(1 - t^{24})^2 \\ &= (1 + t^8)(1 - t^8 + 3t^{16} + 6t^{24} + 5t^{40} + 2t^{48}) \\ &\quad /(1 - t^8)(1 - t^{16})(1 - t^{24})^2.\end{aligned}$$

Remark. The Molien series $\Phi_{G_{1,3}}(t)$ and $\Phi_{G_{1,4}}(t)$ were determined by Runge [22] and Oura [20], respectively.

Finally we describe the invariant rings for these Molien series. We first consider the Hamming weight enumerators of binary Type II codes. In this case, the invariant ring for $G_{1,1}^8$ is generated by the weight enumerators of the extended Hamming $[8,4,4]$ code and the extended Golay $[24,12,8]$ code. Now let us consider complete and symmetrized weight enumerators of Type II codes over $\mathbb{Z}_4$. In [3], the invariant ring for $H_{1,2}^8$ was investigated under the condition that Type II codes contain all-one vector, that is, they investigated the invariant ring for the group K generated by $H_{1,2}^8$ and the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The group K has the same order as $H_{1,2}^8$. Thus the invariant ring for $H_{1,2}^8$ is

$$\mathbb{C}[\phi_8, \phi_8', \phi_{24}] \oplus \phi_{16}\mathbb{C}[\phi_8, \phi_8', \phi_{24}]$$

where $\phi_8, \phi_8', \phi_{16}$ and ϕ_{24} are the symmetrized weight enumerators of Type II codes $O_8, Q_8, RM(1,4)+2RM(2,4)$ and the lifted Golay code G_{24} over $\mathbb{Z}_4$. For the complete weight enumerators, a Magma computation shows that the invariant ring of $G_{1,2}^8$ has the homogenous system of parameters of degrees $8, 8, 8$ and 24. This means that the invariant ring has exactly the Molien series of the form

$$\frac{1+t^8+2t^{16}+2t^{24}+t^{32}+t^{40}}{(1-t^8)^3(1-t^{24})}.$$

Let $\mathcal{W}(n)$ be the ring generated by the g-th complete weight enumerators of Type II codes of length n. We have verified by computer that $\dim \mathcal{W}(8) = 4$ and $\dim \mathcal{W}(16) = 11$ however we have checked only $\dim \mathcal{W}(24) \geq 23$. Thus it is not known if the invariant ring for $G_{1,2}^8$ is generated by the complete weight enumerators of Type II codes over $\mathbb{Z}_4$.

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