HARMONIC TUTTE POLYNOMIALS OF MATROIDS

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Abstract. In the present paper, we introduce the concept of harmonic Tutte polynomials of matroids and discuss some of their properties. In particular, we generalize Greene's theorem, thereby expressing harmonic weight enumerators of codes as evaluations of harmonic Tutte polynomials.

Key Words: Tutte polynomials, weight enumerators, matroids, codes, harmonic functions.

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1. Introduction

Construction of the matroids from various algebraic structures, like graphs and matrices (see [24]), is one of the most important problems in the theory of matroids. It is Crapo [11] who pointed out this problem by extending the definition of Tutte polynomial $T(M;x,y)$ to a matroid M which was introduced by Tutte [27, 28, 29] for a graph. Later, Greene [13] proved a remarkable connection between the weight enumerator $W_C(x,y)$ of an $[n,k]$ code C and the Tutte polynomial of its matroid M_C; this identity, which is known as Greene identity, is as follows:

$$W_C(x,y) = (x-y)^k y^{n-k} T\left(M_C; \frac{x+(q-1)y}{x-y}, \frac{x}{y}\right).$$

As an application of the above relation, Greene [13] also gave an alternative proof of the MacWilliams identity for the weight enumerator of an $[n,k]$ code.

Delsarte [12] introduced the concept of discrete harmonic function on a finite set. In the study of coding theory, Bachoc [2, 3] associated the discrete harmonic function to codes by introducing the concept of harmonic weight enumerator $W_{C,f}$ of a code C, and gave a striking

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generalization of the MacWilliams identity. Tanabe [26] studied the $\mathbb{F}_q$–analogue of the harmonic weight enumerators of codes.

In this paper, we introduce the notion of the harmonic Tutte polynomials associated with a harmonic function of a certain degree, and give a Greene-type identity which we call the *generalized Greene identity*, which relates the harmonic weight enumerator of a code and the harmonic Tutte polynomial of the matroid corresponding to the code. Moreover, as an application of the generalized Greene identity, we give a combinatorial proof of Bachoc's MacWilliams-type identity which is stated in Theorem 2.1.

This paper is organized as follows: In Section 2, we studied most of the basic definitions and properties in coding theory and matroid theory used in this paper. In Section 3, we define the harmonic Tutte polynomial, and obtain a relation between the harmonic Tutte polynomials of a matroid and its dual (Theorem 3.1). Moreover, we reinterpret the definition of harmonic weight enumerators of codes (Theorem 3.2). In Section 4, we give a generalization of Greene identity (Theorem 4.1) with an application in the proof of the MacWilliams identity for harmonic weight enumerator. Finally, in Section 5, we conclude the paper with some remarks.

2. Basic definitions and notions

In this section, we give some basic definitions and properties of codes and matroids that are necessary for this paper. We follow [14, 17, 23, 24, 26] for the discussions. Moreover, we recall some definitions and properties of the (discrete) harmonic functions; see [2, 12, 26] for more detail.

2.1. **Discrete harmonic functions.** Let $E := \{1, 2, \dots, n\}$ be a finite set of first n positive integers. Let 2^E denote the set of all subsets of E. We define $E_d := \{X \in 2^E \mid |X| = d\}$ for $d = 0, 1, \dots, n$. We denote by $\mathbb{R}2^E$, $\mathbb{R}E_d$ the real vector spaces spanned by the elements of 2^E, E_d, respectively. An element of $\mathbb{R}E_d$ is denoted by

$$f := \sum_{Z \in E_d} f(Z)Z \tag{1}$$

and is identified with the real-valued function on E_d given by $Z \mapsto f(Z)$. Such an element $f \in \mathbb{R}E_d$ can be extended to an element $\widetilde{f} \in \mathbb{R}2^E$ by setting, for all $X \in 2^E$,

$$\widetilde{f}(X) := \sum_{Z \in E_d, Z \subset X} f(Z). \tag{2}$$

If an element $g \in \mathbb{R}2^E$ is equal to some $\widetilde{f}$, for $f \in \mathbb{R}E_d$, we say that g has degree d. We call the vector space $\mathbb{R}E_d$ the *homogeneous space* of degree d, and denoted by $\mathrm{Hom}_d(n)$. The differentiation γ is the operator defined by the linear form

$$\gamma(Z) := \sum_{Y \in E_{d-1}, Y \subset Z} Y \tag{3}$$

for all $Z \in E_d$ and for all $d = 0, 1, \dots n$, and $\mathrm{Harm}_d(n)$ is the kernel of γ:

$$\mathrm{Harm}_d(n) := \ker\left(\gamma\big|_{\mathbb{R}E_d}\right).$$

Remark 2.1 ([2, 12])**.** Let $f \in \mathrm{Harm}_d(n)$. Then $\gamma^{d-i}(f) = 0$ for all $0 \le i \le d-1$. That is, from (3) $\sum_{Z \in E_d, X \subset Z} f(Z) = 0$ for any $X \in E_i$.

Remark 2.2. Let $f \in \mathrm{Harm}_d(n)$. Since $\sum_{Z \in E_d} f(Z) = 0$, then it is easy to check from (3) that $\sum_{X \in E_t} \widetilde{f}(X) = 0$, where $1 \le d \le t \le n$.

2.2. **Linear codes.** Let $\mathbb{F}_q$ be a finite field of order q, where q is a prime power. Then $V := \mathbb{F}_q^n$ denotes the vector space of dimension n with ordinary inner product:

$$\mathbf{u} \cdot \mathbf{v} := u_1 v_1 + \cdots + u_n v_n$$

for $\mathbf{u}, \mathbf{v} \in V$, where $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$. Let $\mathrm{supp}(\mathbf{u}) := \{i \in E \mid u_i \neq 0\}$ and $\mathrm{wt}(\mathbf{u}) := |\mathrm{supp}(\mathbf{u})|$ for $\mathbf{u} \in V$. Let $V_d := \{\mathbf{u} \in V \mid \mathrm{wt}(\mathbf{u}) = d\}$. An element $f \in \mathbb{R}E_d$ can be extended to an element $\acute{f} \in \mathbb{R}V$ by setting, for all $\mathbf{u} \in V$,

$$\acute{f}(\mathbf{u}) := \sum_{\substack{\mathbf{v} \in V_d, \\ \mathrm{supp}(\mathbf{v}) \subset \mathrm{supp}(\mathbf{u})}} f(\mathbf{v}). \tag{4}$$

An $\mathbb{F}_q$–*linear code* of length n is a linear subspace of V. An $\mathbb{F}_q$–linear code of length n with dimension k, is called an $[n, k]$ linear code. Let C be an $\mathbb{F}_q$–linear code. We denote by $C^\perp$ the *dual code* of C and defined as:

$$C^\perp := \{\mathbf{u} \in V \mid \mathbf{u} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{v} \in C\}.$$

The *weight distribution* of C is the sequence $\{A_i \mid i = 0, 1, \dots, n\}$, where A_i is the number of codewords of weight i. The polynomial

$$W_C(x, y) := \sum_{\mathbf{u} \in C} x^{n-\mathrm{wt}(\mathbf{u})} y^{\mathrm{wt}(\mathbf{u})} = \sum_{i=0}^{n} A_i x^{n-i} y^i$$

is called the *weight enumerator* of C and satisfies the MacWilliams identity:

$$W_{C^\perp}(x,y) = \frac{1}{|C|} W_C(x+(q-1)y, x-y).$$

Bachoc [2] introduced the concept of harmonic weight enumerator for a binary code which was later defined for an $\mathbb{F}_q$–linear code by Tanabe [26] as follows.

Definition 2.1. Let C be an $\mathbb{F}_q$–linear code of length n. Let $f \in \mathrm{Harm}_d(n)$. The *harmonic weight enumerator* associated with C and f is

$$W_{C,f}(x,y) := \sum_{\mathbf{u} \in C} \acute{f}(\mathbf{u}) x^{n-\mathrm{wt}(\mathbf{u})} y^{\mathrm{wt}(\mathbf{u})}.$$

Theorem 2.1 ([26], MacWilliams type identity)**.** *Let $W_{C,f}(x,y)$ be the harmonic weight enumerator of an $\mathbb{F}_q$–linear code C associated to $f \in \mathrm{Harm}_d(n)$. Then*

$$W_{C,f}(x,y) = (xy)^d Z_{C,f}(x,y),$$

where $Z_{C,f}$ is a homogeneous polynomial of degree $n-2d$, and satisfies

$$Z_{C^\perp,f}(x,y) = (-1)^d \frac{q^{n/2}}{|C|} Z_{C,f}\left(\frac{x+(q-1)y}{\sqrt{q}}, \frac{x-y}{\sqrt{q}}\right).$$

Let C be an $\mathbb{F}_q$–linear code of length n and let $f \in \mathrm{Harm}_d(n)$. Then the weight distribution of C associated to f is defined as $A_{i,f} := \sum_{\mathbf{u} \in C, \mathrm{wt}(\mathbf{u})=i} \acute{f}(\mathbf{u})$. Therefore the harmonic weight enumerator of C associated with f can be rewrite as

$$W_{C,f}(x,y) = \sum_{i=0}^{n} A_{i,f} x^{n-i} y^i.$$

Now from the above definition we have by Theorem 2.1,

$$Z_{C,f}(x,y) = \sum_{i=0}^{n} A_{i,f} x^{n-i-d} y^{i-d}.$$

Remark 2.3. If $\deg f = 0$, we have $A_{i,f} = A_i$, that is, $W_{C,f}(x,y)$ becomes the usual weight enumerator $W_C(x,y)$.

2.3. **Matroids.** The matroids can be defined in several equivalent ways. We prefer the definition which is in terms of independent sets. A (finite) *matroid* M is an ordered pair $(E, \mathcal{I})$ consisting of set E and $\mathcal{I}$ is the collection of subsets of E satisfying the following conditions:

(M1) $\emptyset \in \mathcal{I}$,
(M2) if $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$, and

(M3) if $I, J \in \mathcal{I}$ with $|I| < |J|$, then there exists $j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called the *independent* sets of M, and E is called the *ground set* of M. A subset of the ground set E that does not belongs to $\mathcal{I}$ is called *dependent*. An independent set I is called a *basis* if it becomes dependent on adding any element of $E \setminus I$.

It follows from axiom (M3) that the cardinalities of all bases in a matroid M are equal; this cardinality is called the *rank* of M. The *rank* $\rho(J)$ of an arbitrary subset J of E is the size of the largest independent subset of J. That is, $\rho(J) := \max\{|I| : I \in \mathcal{I} \text{ and } I \subset J\}$. In particular, $\rho(\emptyset) = 0$. We call $\rho(E)$ the rank of M. We refer the readers to [24] for detailed discussion.

Definition 2.2. Let M be a matroid on the set E having a rank function ρ. The *Tutte polynomial* of M is defined as follows:

$$T(M; x, y) := \sum_{J \subset E} (x-1)^{\rho(E)-\rho(J)}(y-1)^{|J|-\rho(J)}.$$

Definition 2.3. Let A be a $k \times n$ matrix over a finite field $\mathbb{F}_q$. This gives a matroid M on the set E in which a set $\mathcal{I}$ is independent if and only if the family of columns of A whose indices belong to $\mathcal{I}$ is linearly independent. Such a matroid is called a *vector matroid*. In the subsequent sections of this note, by matroids we mean vector matroids.

For an $\mathbb{F}_q$–linear code C, M_C denotes the vector matroid that corresponds to C. We next recall this construction, which is treated in [15]. Let G be a $k \times n$ matrix with rank k over the finite field $\mathbb{F}_q$. The set E is indexing the columns of G. Let $\mathcal{I}_G$ be the collection of all subsets J of E such that the submatrix G_J consisting of the columns of G at the positions of J are independent. Then $M_G := (E, \mathcal{I}_G)$ is a matroid. If G_1 and G_2 are generator matrices of an $\mathbb{F}_q$–linear code C, then $(E, \mathcal{I}_{G_1}) = (E, \mathcal{I}_{G_2})$. Therefore, the matroid $M_C := (E, \mathcal{I}_C)$ of an $\mathbb{F}_q$-linear code C is well defined by $(E, \mathcal{I}_G)$ for some generator matrix G of C.

3. Harmonic generalizations of polynomials

3.1. **Harmonic Tutte Polynomials.** In this section, we define the Tutte polynomials of a (finite) matroid M associated with a harmonic function. We also present a very useful relation between the Tutte polynomial of a matroid and its dual associated to a harmonic function.

Definition 3.1. Let $M = (E, \mathcal{I})$ be a matroid with rank function ρ, and $f \in \mathrm{Hom}_d(n)$ be a real-valued function of degree d. Then the *weighted Tutte polynomial* of M associated to f is defined as follows:

$$T(M, f; x, y) := \sum_{J \subset E} \widetilde{f}(J)(x-1)^{\rho(E)-\rho(J)}(y-1)^{|J|-\rho(J)}.$$

In particular, if $f \in \mathrm{Harm}_d(n)$, then we call the weighted Tutte polynomial $T(M, f; x, y)$ the *harmonic Tutte polynomial* associated with f.

We define

$$\mathcal{I}^{\perp} := \{I \in 2^E \mid I \subset E \setminus A \text{ for some } A \in \mathcal{B}(M)\},$$

where $\mathcal{B}(M)$ be the collection of all bases of M. Then $M^{\perp} := (E, \mathcal{I}^{\perp})$ is called the *dual matroid* of M. It is well known that if ρ is the rank function of a matroid $M = (E, \mathcal{I})$, then the rank function of $M^{\perp} = (E, \mathcal{I}^{\perp})$ is given as follows: for any $J \subset E$,

$$\rho^{\perp}(J) := |J| + \rho(E \setminus J) - \rho(E)$$

(see[24]). In particular, $\rho^{\perp}(E) + \rho(E) = |E|$. The correspondence between the harmonic Tutte polynomial of a matroid M and its dual $M^{\perp}$ associated to a harmonic function is given as follows:

Theorem 3.1. *Let $M = (E, \mathcal{I})$ be a matroid with a rank function ρ, and $f \in \mathrm{Harm}_d(n)$. Then $T(M^{\perp}, f; x, y) = (-1)^d T(M, f; y, x)$.*

Before giving a proof of the above theorem, we need to know about the following technical lemma on harmonic functions from [2].

Lemma 3.1 ([2])**.** *Let $f \in \mathrm{Harm}_d(n)$ and $J \subset E$. Let*

$$f^{(i)}(J) := \sum_{\substack{Z \in E_d, \\ |J \cap Z| = i}} f(Z).$$

Then for all $0 \leq i \leq d$, $f^{(i)}(J) = (-1)^{d-i}\binom{d}{i}\widetilde{f}(J)$.

Remark 3.1. From the definition of $\widetilde{f}$ for $f \in \mathrm{Harm}_d(n)$, we have $\widetilde{f}(J) = 0$ for any $J \in 2^E$ such that $|J| < d$. Let $I, J \in 2^E$ such that $I = E \setminus J$. Then

$$\widetilde{f}(J) = \sum_{\substack{Z \in E_d, \\ Z \subset J}} f(Z) = \sum_{\substack{Z \in E_d, \\ |Z \cap I| = 0}} f(Z) = f^{(0)}(I) = (-1)^d \widetilde{f}(E \setminus J).$$

We have from the above equality that if $|J| > n - d$, then $\widetilde{f}(J) = 0$.

Proof of Theorem 3.1. Let M be a matroid on E with rank function ρ. Then $M^{\perp}$ is the dual matroid of M with rank function $\rho^{\perp}(J) =$

$|J| + \rho(E \setminus J) - \rho(E)$ for any $J \subset E$. Therefore,

$$\begin{aligned}
T(M^{\perp}, f; x, y) &= \sum_{J \subset E} \widetilde{f}(J)(x-1)^{\rho^{\perp}(E)-\rho^{\perp}(J)}(y-1)^{|J|-\rho^{\perp}(J)} \\
&= \sum_{J \subset E} \widetilde{f}(J)(x-1)^{\rho^{\perp}(E)-|J|-\rho(E\setminus J)+\rho(E)}(y-1)^{|J|-|J|-\rho(E\setminus J)+\rho(E)} \\
&= \sum_{J \subset E} \widetilde{f}(J)(x-1)^{|E\setminus J|-\rho(E\setminus J)}(y-1)^{\rho(E)-\rho(E\setminus J)} \\
&= (-1)^{d} \sum_{J \subset E} \widetilde{f}(E \setminus J)(y-1)^{\rho(E)-\rho(E\setminus J)}(x-1)^{|E\setminus J|-\rho(E\setminus J)} \\
&= (-1)^{d} T(M, f; y, x).
\end{aligned}$$

This completes the proof. □

3.2. **Harmonic Weight Enumerator.** In this section, we introduce a new approach to define the harmonic weight enumerators of an $\mathbb{F}_q$–linear code. This formulation is inspired from Jurrius and Pellikaan [15].

Definition 3.2. Let E be a finite set of cardinality n. Again let C be an $\mathbb{F}_q$–linear code of length n. Then for an arbitrary subset $J \subset E$, we define

$$\begin{aligned}
C(J) &:= \{\mathbf{c} \in C \mid c_j = 0 \text{ for all } j \in J\} \\
\ell(J) &:= \dim C(J) \\
B_J &:= q^{\ell(J)} - 1.
\end{aligned}$$

Lemma 3.2 ([15])**.** *Let C be an $[n, k]$ linear code with generator matrix G. Assume that the columns of G is indexed by the set E. Let G_J be the $k \times t$ submatrix of G consisting of the columns of G indexed by $J \in E_t$, and let $\rho(J)$ be the rank of G_J. Then $\ell(J) = k - \rho(J)$.*

Now we have the following proposition.

Proposition 3.1. *Let $f \in \mathrm{Harm}_d(n)$ and $J \subset E$. Define*

$$B_{t,f} := \sum_{J \in E_t} \widetilde{f}(J) B_J.$$

Then we have the following relation between $B_{t,f}$ and $A_{i,f}$ as follows:

$$B_{t,f} = (-1)^{d} \sum_{i=d}^{n-t} \binom{n-d-i}{t-d} A_{i,f},$$

if $d \leq t \leq n-d$; otherwise $B_{t,f} = 0$.

Proof. It is immediate from Remark 3.1 that $B_{t,f} = 0$ for $0 \le t < d$ and $n-d < t \le n$. We now focus on t with $d \le t \le n-d$. From the construction of $B_{t,f}$ and Remark 3.1, it is clear that

$$B_{t,f} = (-1)^d \binom{t}{d}^{-1} \sum_{\substack{J \in E_t, X \in E_d,\\ X \subset J}} \widetilde{f}(E \setminus J) B_J.$$

Therefore, it is sufficient to show that for $d \le t \le n-d$,

$$\binom{t}{d}^{-1} \sum_{\substack{J \in E_t, X \in E_d,\\ X \subset J}} \widetilde{f}(E \setminus J) B_J = \sum_{i=d}^{n-t} \binom{n-d-i}{t-d} A_{i,f}.$$

Now following the definition of B_J, we can easily observe that

$$\begin{aligned}\sum_{\substack{J \in E_t, X \in E_d,\\ X \subset J}} \widetilde{f}(E \setminus J) B_J &= \sum_{\substack{J \in E_t, X \in E_d,\\ X \subset J}} \sum_{\substack{\mathbf{c} \in C,\\ \mathrm{supp}(\mathbf{c}) \cap J = \emptyset,\\ \mathbf{c} \neq 0}} \widetilde{f}(E \setminus J)\\ &= \sum_{\substack{\mathbf{c} \in C,\\ \mathbf{c} \neq 0}} \sum_{\substack{J \in E_t, X \in E_d,\\ \mathrm{supp}(\mathbf{c}) \cap J = \emptyset,\\ X \subset J}} \widetilde{f}(E \setminus J).\end{aligned}$$

Note that from the definition of harmonic function and Remark 2.1 we have $\sum_{\substack{J \in E_t, X \in E_d,\\ \mathrm{supp}(\mathbf{c}) \cap J = \emptyset,\\ X \subset J}} \widetilde{f}(E \setminus J) = 0$ for $\mathbf{c} \in C$ with $\mathrm{wt}(\mathbf{c}) < d$. Therefore,

$$\sum_{\substack{J \in E_t, X \in E_d,\\ X \subset J}} \widetilde{f}(E \setminus J) B_J = \sum_{i=d}^{n-t} \sum_{\substack{\mathbf{c} \in C,\\ \mathrm{wt}(\mathbf{c}) = i}} \sum_{\substack{J \in E_t, X \in E_d,\\ \mathrm{supp}(\mathbf{c}) \cap J = \emptyset,\\ X \subset J}} \widetilde{f}(E \setminus J).$$

After expanding the right hand side of the expression above for each $\mathbf{c} \in C$ with $\mathrm{wt}(\mathbf{c}) = i$ for $d \le i \le n-t$, Remark 2.1 together with (2) implies that

$$\sum_{\substack{J \in E_t, X \in E_d,\\ X \subset J}} \widetilde{f}(E \setminus J) B_J = \sum_{i=d}^{n-t} \sum_{\substack{\mathbf{c} \in C,\\ \mathrm{wt}(\mathbf{c}) = i}} \binom{t}{d} N_X(J) \acute{f}(\mathrm{supp}(\mathbf{c})),$$

where $N_X(J)$ denotes the number of $J \in E_t$ disjoint from $\mathrm{supp}(\mathbf{c})$ and containing $X \in E_d$. Therefore

$$\sum_{\substack{J \in E_t, X \in E_d,\\ X \subset J}} \widetilde{f}(E \setminus J) B_J = \sum_{i=d}^{n-t} \sum_{\substack{\mathbf{c} \in C,\\ \mathrm{wt}(\mathbf{c}) = i}} \binom{t}{d} \binom{n-d-i}{t-d} \acute{f}(\mathrm{supp}(\mathbf{c})).$$

This completes the proof. $\square$

Now we have the following result.

Theorem 3.2. *Let C be an $\mathbb{F}_q$–linear code of length n, and $f \in \mathrm{Harm}_d(n)$. Then*

$$Z_{C,f}(x,y) = (-1)^d \sum_{t=d}^{n-d} B_{t,f}(x-y)^{t-d}y^{n-t-d}.$$

Proof. By using Proposition 3.1 and using the binomial expansion of $x^{n-i} = ((x-y)+y)^{n-i}$ we have

$$\begin{aligned}
&(-1)^d \sum_{t=d}^{n-d} B_{t,f}(x-y)^{t-d}y^{n-t-d} \\
&= \sum_{t=d}^{n-d}\sum_{i=d}^{n-t} \binom{n-d-i}{t-d} A_{i,f}(x-y)^{t-d}y^{n-t-d} \\
&= \sum_{i=d}^{n-d} A_{i,f}\left(\sum_{t=d}^{n-i} \binom{n-d-i}{t-d}(x-y)^{t-d}y^{(n-d-i)-(t-d)}\right)y^{i-d} \\
&= \sum_{i=d}^{n-d} A_{i,f}x^{n-d-i}y^{i-d} \\
&= \sum_{i=0}^{n} A_{i,f}x^{n-i-d}y^{i-d} \\
&= Z_{C,f}(x,y),
\end{aligned}$$

since $A_{i,f} = 0$ for $i < d$ and $i > n-d$. $\square$

3.3. **Examples.** We assume that $E = \{1,2,3\}$. Let

$$\mathrm{Harm}_1(3) \ni f = a\{1\} + b\{2\} - (a+b)\{3\}$$

be a (discrete) harmonic function of degree 1, where $f(\{1\}) = a$, $f(\{2\}) = b$ and $f(\{3\}) = -(a+b)$. Then

$$\begin{array}{llll}
\widetilde{f}(\emptyset) = 0, & \widetilde{f}(\{1\}) = a, & \widetilde{f}(\{2\}) = b, & \widetilde{f}(\{3\}) = -(a+b) \\
\widetilde{f}(\{1,2\}) = a+b, & \widetilde{f}(\{1,3\}) = -b, & \widetilde{f}(\{2,3\}) = -a, & \widetilde{f}(\{1,2,3\}) = 0.
\end{array}$$

Let C be a $[3,2]$ code over $\mathbb{F}_2$ with generator matrix as follows:

$$G = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The columns of the matrix G is indexed by E. Then the matroid corresponding to C is $M_C = (E, \mathcal{I})$, where

$$\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\}.$$

Here $\rho(E) = 2$. It is easy to find the ranks of all the subsets of E.

By direct calculation, we have

$$\begin{aligned} T(M_C, f; x, y) &= \sum_{J \subset E} \widetilde{f}(J)(x-1)^{\rho(E)-\rho(J)}(y-1)^{|J|-\rho(J)} \\ &= (a+b)(x-1)(y-1) - (a+b). \end{aligned}$$

The elements of C are listed as follows:

$$(0,0,0), (0,0,1), (1,1,0), (1,1,1).$$

The harmonic weight enumerator of C associated to f is

$$W_{C,f} = -(a+b)x^2y + (a+b)xy^2 = (xy)^1 Z_{C,f},$$

where $Z_{C,f} = (a+b)(y-x)$.

4. Generalized Greene's Identity

Let M_C be a matroid associated to an $\mathbb{F}_q$–linear code C of length n. It is immediate from [10, 13, 15] that $M_C^{\perp} = M_{C^{\perp}}$. Now we have the following proposition.

Proposition 4.1. *Let C be an $[n,k]$ code and M_C be its matroid. Let f be a harmonic function of degree d. Then*

$$T(M_C, f; x, y) = \sum_{t=d}^{n-d} \sum_{J \in E_t} \tilde{f}(J)(x-1)^{\ell(J)}(y-1)^{\ell(J)-(k-t)}.$$

Proof. The proposition follows from $\ell(J) = k - \rho(J)$ for any $J \in E_t$ by Lemma 3.2, and $\rho(E) = k$. □

Now we have the following harmonic generalization of the Greene's identity.

Theorem 4.1. *Let C be an $[n,k]$ code and f be a harmonic function with degree d. Then*

$$Z_{C,f}(x,y) = (-1)^d (x-y)^{k-d} y^{n-k-d} T\left(M_C, f; \frac{x+(q-1)y}{x-y}, \frac{x}{y}\right).$$

Proof. By using the Proposition 4.1 and Remark 2.2, we can write

$$\begin{aligned}
T\left(M_C, f; \frac{x+(q-1)y}{x-y}, \frac{x}{y}\right) &= \sum_{t=d}^{n-d} \sum_{J \in E_t} \widetilde{f}(J) \left(\frac{qy}{x-y}\right)^{\ell(J)} \left(\frac{x-y}{y}\right)^{\ell(J)-(k-t)} \\
&= \sum_{t=d}^{n-d} \sum_{J \in E_t} \widetilde{f}(J) q^{\ell(J)} \left(\frac{y}{x-y}\right)^{\ell(J)} \left(\frac{x-y}{y}\right)^{\ell(J)-(k-t)} \\
&= \sum_{t=d}^{n-d} \sum_{J \in E_t} \widetilde{f}(J)((q^{\ell(J)}-1)+1)(x-y)^{-(k-t)} y^{k-t} \\
&= \sum_{t=d}^{n-d} \sum_{J \in E_t} \widetilde{f}(J)(B_J+1)(x-y)^{-(k-t)} y^{k-t} \\
&= \sum_{t=d}^{n-d} \left(\sum_{J \in E_t} \widetilde{f}(J) B_J + \sum_{J \in E_t} \widetilde{f}(J)\right)(x-y)^{-(k-t)} y^{k-t} \\
&= \sum_{t=d}^{n-d} \left(B_{t,f}+0\right)(x-y)^{-(k-t)} y^{k-t} \\
&= \sum_{t=d}^{n-d} B_{t,f}(x-y)^{-(k-t)} y^{k-t}.
\end{aligned}$$

Therefore, from Theorem 3.2 we have

$$\begin{aligned}
(-1)^d (x-y)^{k-d} y^{n-k-d} T\left(M_C, f; \frac{x+(q-1)y}{x-y}, \frac{x}{y}\right) \\
&= (-1)^d \sum_{t=d}^{n-d} B_{t,f}(x-y)^{t-d} y^{n-t-d} \\
&= Z_{C,f}(x,y).
\end{aligned}$$

This completes the proof. □

Now we give an alternative proof of the $\mathbb{F}_q$–analogue of Bachoc's MacWilliams type identity (see [2]) stated in Theorem 2.1 as an application of Theorem 4.1.

Proof of Theorem 2.1. Let C be an $[n,k]$ code, and M_C be its matroid. Then

$$\begin{aligned}
&(-1)^d\frac{q^{n/2}}{|C|}Z_{C,f}\left(\frac{x+(q-1)y}{\sqrt{q}},\frac{x-y}{\sqrt{q}}\right)\\
&\quad=(-1)^{2d}\frac{q^{n/2}}{q^k}\left(\frac{qy}{\sqrt{q}}\right)^{k-d}\left(\frac{x-y}{\sqrt{q}}\right)^{n-k-d}T\left(M_C,f;\frac{x}{y},\frac{x+(q-1)y}{x-y}\right)\\
&\quad=(x-y)^{n-k-d}y^{k-d}T\left(M_C,f;\frac{x}{y},\frac{x+(q-1)y}{x-y}\right)\\
&\quad=(-1)^d(x-y)^{(n-k)-d}y^{n-(n-k)-d}T\left(M_{C^\perp},f;\frac{x+(q-1)y}{x-y},\frac{x}{y}\right)\\
&\quad=(-1)^d(x-y)^{\dim C^\perp-d}y^{n-\dim C^\perp-d}T\left(M_{C^\perp},f;\frac{x+(q-1)y}{x-y},\frac{x}{y}\right)\\
&\quad=Z_{C^\perp,f}(x,y).
\end{aligned}$$

Hence Theorem is proved. □

5. Concluding Remarks

We close this paper with the following design theoretical remark that gives an application of the harmonic function connecting the t-designs with matroids.

Remark 5.1. Let n,k,t and λ be non-negative integers such that $n\geq k\geq t$ and $\lambda\geq 1$. A t-(n,k,λ) *design* (in short, t-design) is a pair $\mathcal{D}:=(E,\mathcal{B})$, where E is a finite set of point of cardinality n, and $\mathcal{B}$ is a collection of k-element subsets of E called *blocks*, with the property that any t points are contained in precisely λ blocks. Some properties of combinatorial t-designs obtained from codes were discussed in [1, 2, 4, 8, 9, 16, 20, 21, 22, 25] and their analogies in the theory of lattices and vertex operator algebras were discussed in [4, 5, 6, 7, 18, 19, 20].

The harmonic functions have many applications; particularly, the relations between design theory and coding theory were stated in Bachoc [2]: the set of words with fixed weight in a binary code C forms a t-design if and only if $W_{C,f}(x,y)=0$ for all $f\in\mathrm{Harm}_d(n)$, $1\leq d\leq t$. Then it is trivial that if $T(M_C,f;x,y)=0$ for all $f\in\mathrm{Harm}_d(n)$, $1\leq d\leq t$, then the set of words with fixed weight in an $\mathbb{F}_q$-linear code C forms a t-design.

We will more precisely discuss about a relation between matroids and combinatorial designs with respect to “harmonic Tutte polynomials” in the forthcoming paper.

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