JACOBI POLYNOMIALS AND DESIGN THEORY I

HIMADRI SHEKHAR CHAKRABORTY, TSUYOSHI MIEZAKI,
MANABU OURA, AND YUUHO TANAKA*

Abstract. In this paper, we introduce the notion of Jacobi polynomials of a code with multiple reference vectors, and give the MacWilliams type identity for it. Moreover, we derive a formula to obtain the Jacobi polynomials using the Aronhold polarization operator. Finally, we describe some facts obtained from Type III and Type IV codes that interpret the relation between the Jacobi polynomials and designs.

Key Words: Codes, Jacobi polynomials, designs, invariant theory.

2010 *Mathematics Subject Classification.* Primary 11T71; Secondary 94B05, 11F11.

1. Introduction

A. Bonnecaze et al. [4] took the notion of Jacobi polynomials, a celebrated generalization of weight enumerators [20, 22] that were introduced by M. Ozeki [26] for codes as an analogue to Jacobi forms [2, 16] as a powerful generalization of modular form [15, 27] of Lattices [8]. They gave a formula to compute the Jacobi polynomials of a binary code as an application of combinatorial t-designs using an operator, known as Aronhold polarization operator. Many authors studied the combinatorial t-designs and discussed their properties in [1, 14, 23, 24] that were derived from codes and their analogies. Moreover, P.J. Cameron [7] gave the notion of generalized t-designs and discussed its properties. Furthermore, A. Bonnecaze et al. [4] constructed various types of designs such as group divisible designs, packing designs and covering designs. To establish the relationship between these designs and the Jacobi polynomials, they studied Jacobi polynomials for Type II codes through invariant theory [17, 25].

In this paper, we give the generalizations and analogues of some results in [4]. We define the Jacobi polynomials with multiple reference vectors for codes, and give the MacWilliams type identity for it. As an

*Corresponding author.

analogue of the combinatorial interpretation of the polarization that was given in [4], is given here for codes that holds generalized t-designs for every given weight of the codewords. In addition, we study some Type III (resp. Type IV) codes of specific lengths, and determine the polynomials that generate the space of Jacobi polynomials for a Type III (resp. Type IV) code with respect to reference vectors of a particular length. Moreover, we observe from the examples that the number of blocks of a packing (resp. covering) design correspond to the coefficients in Jacobi polynomials.

This paper is organized as follows. In Section 2, we discuss the basic definitions and properties of codes that needed to understand this paper. In Section 3, we give the MacWilliams type identity (Theorem 3.1) for the Jacobi polynomials of a code with multiple reference vectors. In Section 4, we see how polarization operator acts to obtain the Jacobi polynomials with multiple reference vectors (Theorem 4.2, Theorem 4.3). In Section 5, we disclose some facts between a Type III (resp. Type IV) code of specific length and designs of various kinds with the help of the Jacobi polynomials. Finally, we conclude the paper with some remarks in Section 6.

All computer calculations in this paper were done with the help of Magma [6].

2. Preliminaries

Let $\mathbb{F}_q$ be a finite field of order q, where q is a prime power. Then $\mathbb{F}_q^n$ denotes the vector space of dimension n over $\mathbb{F}_q$. The elements of $\mathbb{F}_q^n$ are known as *vectors*. The *Hamming weight* of $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{F}_q^n$ is denoted by $\mathrm{wt}(\mathbf{u})$ and defined to be the number of i's such that $u_i \neq 0$. Let $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ be the vectors of $\mathbb{F}_q^n$. Then the *inner product* of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}_q^n$ is given by

$$\mathbf{u} \cdot \mathbf{v} := u_1 v_1 + \cdots + u_n v_n.$$

If q is an even power of an arbitrary prime p, then it is convenient to consider another inner product given by

$$\mathbf{u} \cdot \mathbf{v} := u_1 \overline{v_1} + \cdots + u_n \overline{v_n},$$

where $\overline{v_i} := v_i^{\sqrt{q}}$. An $\mathbb{F}_q$-linear code of length n is a vector subspace of $\mathbb{F}_q^n$. The elements of an $\mathbb{F}_q$-linear code are called *codewords*. The *dual code* of an $\mathbb{F}_q$-linear code C of length n is defined by

$$C^\perp := \{\mathbf{v} \in \mathbb{F}_q^n \mid \mathbf{u} \cdot \mathbf{v} = \mathbf{0} \text{ for all } \mathbf{u} \in C\}.$$

An $\mathbb{F}_q$-linear code C is called *self-dual* if $C = C^\perp$. It is well known that the length n of a self-dual code over $\mathbb{F}_q$ is even and the dimension is $n/2$.

To study self-dual codes in detail, we refer the readers to [3, 17, 21, 25]. A self-dual code C over $\mathbb{F}_2$ or $\mathbb{F}_4$ of length $n \equiv 0 \pmod 2$ having even weight is called *Type* I and *Type* IV, respectively. A self-dual code C over $\mathbb{F}_2$ of length $n \equiv 0 \pmod 8$ is called *Type* II if the weight of each codeword of C is multiple of 4. Finally, a self-dual code C over $\mathbb{F}_3$ of length $n \equiv 0 \pmod 4$ is called *Type* III if the weight of each codeword of C is multiple of 3.

Definition 2.1. Let C be an $\mathbb{F}_q$-linear code of length n. We denote by A_i^C the number of codewords in C having Hamming weight i. Then the *weight enumerator* of C is defined as

$$W_C(x, y) := \sum_{\mathbf{u} \in C} x^{n-\mathrm{wt}(\mathbf{u})} y^{\mathrm{wt}(\mathbf{u})} = \sum_{i=0}^{n} A_i^C x^{n-i} y^i.$$

Definition 2.2. Let C be an $\mathbb{F}_q$-linear code of length n. Then the *Jacobi polynomial* attached to a set T of coordinate places of the code C is defined as follows:

$$J_{C,T}(w, z, x, y) := \sum_{\mathbf{u} \in C} w^{m_0(\mathbf{u})} z^{m_1(\mathbf{u})} x^{n_0(\mathbf{u})} y^{n_1(\mathbf{u})},$$

where $T \subseteq [n]$, and $m_i(\mathbf{u})$ is the Hamming composition of $\mathbf{u}$ on T and $n_i(\mathbf{u})$ is the Hamming composition of $\mathbf{u}$ on $[n] \backslash T$.

3. MacWilliams type identity

The MacWilliams type identity for the Jacobi polynomial of an $\mathbb{F}_q$–linear code with one reference vector was given in [26]. In this section, we give the MacWilliams type identity for the Jacobi polynomial of an $\mathbb{F}_q$–linear code with multiple reference vectors.

Definition 3.1. Let C be an $\mathbb{F}_q$–linear code of length n. Then the *Jacobi polynomial* of C with respect to ℓ reference vectors $\mathbf{w}_1, \ldots, \mathbf{w}_\ell \in \mathbb{F}_q^n$ is denoted by $J_{C,\mathbf{w}_1,\ldots,\mathbf{w}_\ell}(\{x_\mathbf{a}\}_{\mathbf{a} \in \mathbb{F}_2^{\ell+1}})$ and defined as

$$J_{C,\mathbf{w}_1,\ldots,\mathbf{w}_\ell}(\{x_\mathbf{a}\}_{\mathbf{a} \in \mathbb{F}_2^{\ell+1}}) := \sum_{\mathbf{u} \in C} \prod_{\mathbf{a} \in \mathbb{F}_2^{\ell+1}} x_\mathbf{a}^{N_\mathbf{a}(\mathbf{u},\mathbf{w}_1,\ldots,\mathbf{w}_\ell)}.$$

Here we denote by $N_\mathbf{a}(\mathbf{u}_1, \ldots, \mathbf{u}_g)$ the number of i such that $\mathbf{a} = (\phi(u_{1i}), \ldots, \phi(u_{gi})) \in \mathbb{F}_2^g$, where $\phi(u_{ji}) = 1$ if $u_{ji} \neq 0$, otherwise $\phi(u_{ji}) = 0$.

Note that if $\ell = 1$, the above definition is completely equivalent to the Jacobi polynomial with one reference vector (Definition 2.2).

Let $\mathbb{F}_q$ be a finite field, where $q = p^f$ for some prime number p. A *character* of $\mathbb{F}_q$ is a homomorphism from the additive group $\mathbb{F}_q$ to the multiplicative group of non-zero complex numbers. We review [10, 22] to introduce some fixed non-trivial characters over $\mathbb{F}_q$. Now let $F(x)$ be a primitive irreducible polynomial of degree f over $\mathbb{F}_p$ and let λ be a root of $F(x)$. Then any element $a \in \mathbb{F}_q$ has a unique representation as:

$$a = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_{f-1}\lambda^{f-1},$$

where $a_i \in \mathbb{F}_p$. For $b \in \mathbb{F}_q$, we define $\chi_b(a) := \zeta_p^{a_0b_0+\cdots+a_{f-1}b_{f-1}}$, where ζ_p is the p-th primitive root $e^{2\pi i/p}$ of unity. When $b \neq 0$, then χ_b is a non-trivial character of $\mathbb{F}_q$. Let χ be a non-trivial character of $\mathbb{F}_q$. Then for any $a \in \mathbb{F}_q$, we have the following property:

$$\sum_{b\in\mathbb{F}_q} \chi(ab) := \begin{cases} q & \text{if} \quad a = 0, \\ 0 & \text{if} \quad a \neq 0. \end{cases}$$

Lemma 3.1 ([22])**.** *Let C be an $\mathbb{F}_q$-linear code of length n. For $\mathbf{v} \in \mathbb{F}_q^n$, define*

$$\delta_{C^\perp}(\mathbf{v}) := \begin{cases} 1 & \textit{if } \mathbf{v} \in C^\perp, \\ 0 & \textit{otherwise.} \end{cases}$$

Then we have the following identity:

$$\delta_{C^\perp}(\mathbf{v}) = \frac{1}{|C|}\sum_{\mathbf{u}\in C} \chi(\mathbf{u}\cdot\mathbf{v}).$$

Now we give the MacWilliams type identity for the Jacobi polynomial of an $\mathbb{F}_q$–linear code with respect to multiple reference vectors.

Theorem 3.1 (MacWilliams Identity)**.** *Let C be an $\mathbb{F}_q$–linear code of length n. Again let χ be a non-trivial character of $\mathbb{F}_q$. Let $J_{C,\mathbf{w}_1,\ldots,\mathbf{w}_\ell}(\{x_\mathbf{a}\}_{\mathbf{a}\in\mathbb{F}_2^{\ell+1}})$ be the Jacobi polynomial of C with respect to the reference vectors $\mathbf{w}_1,\ldots,\mathbf{w}_\ell \in \mathbb{F}_q^n$. Then*

$$\begin{aligned} &J_{C^\perp,\mathbf{w}_1,\ldots,\mathbf{w}_\ell}(\{x_\mathbf{a}\}_{\mathbf{a}\in\mathbb{F}_2^{\ell+1}}) \\ &\quad = \frac{1}{|C|} J_{C,\mathbf{w}_1,\ldots,\mathbf{w}_\ell}\left(\left\{\sum_{b\in\mathbb{F}_q} \chi(a_1b)x_{(\phi(b),\phi(a_2),\ldots,\phi(a_{\ell+1}))}\right\}_{\mathbf{a}\in\mathbb{F}_q^{\ell+1}}\right). \end{aligned}$$

Proof. By Lemma 3.1, we can write

$$
\begin{aligned}
&J_{C^\perp,\mathbf{w}_1,\ldots,\mathbf{w}_\ell}(\{x_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{F}_2^{\ell+1}})\\
&\quad= \sum_{\mathbf{u}\in C^\perp} \prod_{\mathbf{a}\in\mathbb{F}_2^{\ell+1}} x_{\mathbf{a}}^{N_{\mathbf{a}}(\mathbf{u},\mathbf{w}_1,\ldots,\mathbf{w}_\ell)}\\
&\quad= \sum_{\mathbf{v}\in\mathbb{F}_q^n} \delta_{C^\perp}(\mathbf{v}) \prod_{\mathbf{a}\in\mathbb{F}_2^{\ell+1}} x_{\mathbf{a}}^{N_{\mathbf{a}}(\mathbf{v},\mathbf{w}_1,\ldots,\mathbf{w}_\ell)}\\
&\quad= \frac{1}{|C|} \sum_{\substack{\mathbf{u}\in C\\ \mathbf{v}\in\mathbb{F}_q^n}} \chi(\mathbf{u}\cdot\mathbf{v}) \prod_{\mathbf{a}\in\mathbb{F}_2^{\ell+1}} x_{\mathbf{a}}^{N_{\mathbf{a}}(\mathbf{v},\mathbf{w}_1,\ldots,\mathbf{w}_\ell)}\\
&\quad= \frac{1}{|C|} \sum_{\substack{\mathbf{u}\in C\\ \mathbf{v}\in\mathbb{F}_q^n}} \chi(u_1v_1+\cdots+u_nv_n) \prod_{1\le i\le n} x_{(\phi(v_i),\phi(w_{1i}),\ldots,\phi(w_{\ell i}))}\\
&\quad= \frac{1}{|C|} \sum_{\mathbf{u}\in C} \prod_{1\le i\le n} \left\{ \sum_{v_i\in\mathbb{F}_q} \chi(u_iv_i) x_{(\phi(v_i),\phi(w_{1i}),\ldots,\phi(w_{\ell i}))} \right\}\\
&\quad= \frac{1}{|C|} \sum_{\mathbf{u}\in C} \prod_{\mathbf{a}\in\mathbb{F}_q^{\ell+1}} \left(\sum_{b\in\mathbb{F}_q} \chi(a_1b) x_{(\phi(b),\phi(a_2),\ldots,\phi(a_{\ell+1}))} \right)^{N_{\mathbf{a}}(\mathbf{u},\mathbf{w}_1,\ldots,\mathbf{w}_\ell)}\\
&\quad= \frac{1}{|C|} J_{C,\mathbf{w}_1,\ldots,\mathbf{w}_\ell}\left(\left\{ \sum_{b\in\mathbb{F}_q} \chi(a_1b) x_{(\phi(b),\phi(a_2),\ldots,\phi(a_\ell+1))} \right\}_{\mathbf{a}\in\mathbb{F}_q^{\ell+1}} \right).
\end{aligned}
$$

Hence the proof is completed. □

4. Generalized t-designs and Jacobi polynomials

Bonnecaze et al. introduced an operator called *polarization operator* in [4], and using this operator, they gave a formula to evaluate the Jacobi polynomial of a binary code from the weight enumerator of the code. In this section, we give a generalized form of the polarization operation, and present an application of this operator in the evaluation of the Jacobi polynomial of a non-binary code associated to the multiple reference vectors.

First we recall the definition of generalized t-designs from [7] as follows. Let t, k, λ be the integers such that $\lambda > 0$ and $k > t > 0$. Again let $\mathbf{k} := (k_1, \ldots, k_n)$ such that $k = \sum_{i=1}^{n} k_i$, $\mathbf{v} := (v_1, \ldots, v_n)$ such that $v_i \geq k_i$ for all i. Let $\mathbf{X} := (X_1, \ldots, X_n)$, where X_i's are pairwise disjoint sets with $|X_i| = v_i$ for all i and

$$\mathcal{B} \subseteq \binom{X_1}{k_1} \times \cdots \times \binom{X_n}{k_n}.$$

Definition 4.1. A t-$(\mathbf{v}, \mathbf{k}, \lambda)$ *design* or a *generalized t-design* (in short) is a pair $\mathcal{D} := (\mathbf{X}, \mathcal{B})$ with the following property: if $\mathbf{t} := (t_1, \ldots, t_n)$ such that $t = \sum_{i=1}^{n} t_i$ satisfying $0 \leq t_i \leq k_i$ for all i, then for any choice $\mathbf{T} := (T_1, \ldots, T_n)$ with $T_i \in \binom{X_i}{t_i}$ for all i, there are precisely λ members $\mathbf{K} := (K_1, \ldots, K_n) \in \mathcal{B}$ for which $T_i \subseteq K_i$ for all i.

Note that in the case when $\mathbf{k} = (k)$ and $\mathbf{v} = (v)$, this is precisely the definition of a combinatorial t-(v, k, λ) design or a *t-design* (in short). We can construct the generalized t-designs from codes as follows.

Let $\mathbf{v} = (v_1, \ldots, v_\ell)$ such that $\sum_{i=1}^{\ell} v_i = n$ and $\mathbf{X} = (X_1, \ldots, X_\ell)$ of pairwise disjoint sets $X_i \subseteq [n]$ with $|X_i| = v_i$. Again let $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{F}_q^n$. Then for $X \subseteq [n]$, we define

$$\begin{aligned} \operatorname{supp}_X(\mathbf{u}) &:= \{i \in X \mid u_i \neq 0\}, \\ \mathbf{K}(\mathbf{u}) &:= (\operatorname{supp}_{X_1}(\mathbf{u}), \ldots, \operatorname{supp}_{X_\ell}(\mathbf{u})), \\ \operatorname{wt}_X(\mathbf{u}) &:= |\operatorname{supp}_X(\mathbf{u})|. \end{aligned}$$

Again for any positive integer k, let $\mathbf{k} = (k_1, \ldots, k_\ell)$ such that $\sum_{i=1}^{\ell} k_i = k$. Let C be an $\mathbb{F}_q$–linear code of length n. Then

$$\begin{aligned} C_{\mathbf{k}} &:= \{\mathbf{u} \in C \mid \operatorname{wt}_{X_i}(\mathbf{u}) = k_i \text{ for all } i\}, \\ \mathcal{B}(C_{\mathbf{k}}) &:= \{\mathbf{K}(\mathbf{u}) \mid \mathbf{u} \in C_{\mathbf{k}}\}. \end{aligned}$$

In general, $\mathcal{B}(C_{\mathbf{k}})$ is a multi-set. We say $C_{\mathbf{k}}$ is a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design if $(\mathbf{X}, \mathcal{B}(C_{\mathbf{k}}))$ is a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design. We say a code is *generalized t-homogeneous* if the codewords of every given weight k hold a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design.

From the above discussion we have the following result. We omit the proof of the theorem since it follows from the above definitions.

Theorem 4.1. *Let C be an $\mathbb{F}_q$-linear code of length n. Let t, k, λ be the integers such that $\lambda > 0$ and $k > t > 0$. Let $\mathbf{v} := (v_1, \ldots, v_\ell)$ such that $v_1 + \cdots + v_\ell = n$. Let $\mathbf{X} := (X_1, \ldots, X_\ell)$ of pairwise disjoint set $X_1, \ldots, X_\ell \subseteq [n]$ with $|X_1| = v_1, \ldots, |X_\ell| = v_\ell$. Let $\mathbf{k} := (k_1, \ldots, k_\ell)$ such that $k_1 + \cdots + k_\ell = k$. Then the set of codewords of C form a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design for every given weight k with $\mathbf{t} = (t_1, \ldots, t_\ell)$ such $t_1 + \cdots + t_\ell = t$ satisfying $0 \leq t_i \leq k_i$ for all i if and only if the Jacobi polynomial $J_{C, \mathbf{w}_1, \ldots, \mathbf{w}_\ell}$ of C associated to the reference vectors $\mathbf{w}_1, \ldots, \mathbf{w}_\ell \in \mathbb{F}_q^n$ such that $\operatorname{wt}_{X_i}(\mathbf{w}_i) = t_i = \operatorname{wt}(\mathbf{w}_i)$ for all i, is invariant.*

The situation described in the above theorem interprets that the Jacobi polynomial $J_{C,\mathbf{w}_1,\ldots,\mathbf{w}_\ell}$ is independent of the choices of the reference vectors $\mathbf{w}_1,\ldots,\mathbf{w}_\ell \in \mathbb{F}_q^n$. In this case, we prefer to denote the Jacobi polynomial as $J_{C,t_1,\ldots,t_\ell}$. In particular, when $\mathbf{k}=(k)$ and $\mathbf{t}=(t)$, it becomes the Jacobi polynomial $J_{C,t}$ as in [4].

Let C be an $\mathbb{F}_q$-linear code of length n. Then the code $C-i$ obtained from C by *puncturing* at coordinate place i. Now from [22] we have the following lemma.

Lemma 4.1. *Let C be a code of length n. Then*

$$W_{C-i}(x,y)=\frac{1}{n}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)W_C(x,y).$$

Let $P(x_0,x_1)$ be a homogeneous polynomial with indeterminates x_0 and x_1. Again let P'_{x_0}(resp. P'_{x_1}) denote the partial derivative with respect to variable x_0 (resp. x_1). Define the polarization operator A_j for any integer $1\le j\le \ell$ as follows:

$$A_j\cdot P(w_j,z_j,x_0,x_1):=w_jP'_{x_0}(x_0,x_1)+z_jP'_{x_1}(x_0,x_1). \tag{4.1}$$

Here the indeterminates in the above equation denote x_a for some $a\in\mathbb{F}_2^{\ell+1}$ as follows: $w_j := x_{(0,0,\ldots,0,\underset{(j+1)\text{th}}{1},0,\ldots,0)}$, $z_j := x_{(1,0,\ldots,0,\underset{(j+1)\text{th}}{1},0,\ldots,0)}$, $x_0:=x_{(0,0,\ldots,0)}$, and $x_1:=x_{(1,0,\ldots,0)}$. Now we have a generalization of [4, Theorem 3] as follows.

Theorem 4.2. *Every code C is a generalized 1-homogenous if and only if for any ℓ-tuple $(0,\ldots,0,1,0,\ldots,0)$ having a single non-zero coordinate, say j-th coordinate with 1, we have*

$$J_{C,0,\ldots,0,1,0,\ldots,0}=\frac{1}{n}A_j\cdot W_C. \tag{4.2}$$

Proof. Let C be generalized 1-homogeneous. Then by Lemma 4.1 one can easily find Equation (4.2) is true. Conversely, the hypothesis implies that the Jacobi polynomial $J_{C,0,\ldots,0,1,0,\ldots,0}$ is uniquely determined. Therefore, by Theorem 4.1 we can say that the codewords of every given weight of C form a generalized 1-design. Hence C is generalized 1-homogeneous. □

Theorem 4.3. *If C is generalized t-homogeneous and contains no codeword of weight $\le t$ then for $\mathbf{t}=(t_1,\ldots,t_\ell)$ such that $t_1+\cdots+t_\ell=t$ we have*

$$J_{C,t_1,\ldots,t_\ell}=\frac{1}{n(n-1)\cdots(n-t+1)}A_\ell^{t_\ell}\cdots A_1^{t_1}\cdot W_C.$$

Proof. The statement is true for $t = 1$ by Theorem 4.2. For $\mathbf{d} := (d_1, \ldots, d_\ell)$ such that $d_1 + \cdots + d_\ell = d < t$ satisfying $0 \le d_i \le t_i$ for all i, let us suppose that

$$J_{C,d_1,\ldots,d_\ell} = \frac{1}{n(n-1)\cdots(n-d+1)} A_\ell^{d_\ell} \cdots A_1^{d_1} \cdot W_C.$$

Let $d_j < t_j$. Then we have

$$\begin{aligned} J_{C,d_1,\ldots,d_{j-1},(d_j+1),d_{j+1},\ldots,d_\ell} &= \frac{1}{n-d} A_j \cdot J_{C,d_1,\ldots,d_{j-1},d_j,d_{j+1},\ldots,d_\ell} \\ &= \frac{1}{n-d} A_j \frac{1}{n(n-1)\cdots(n-d+1)} \\ &\qquad A_\ell^{d_\ell} \cdots A_{j+1}^{d_{j+1}} A_j^{d_j} A_{j-1}^{d_{j-1}} \cdots A_1^{d_1} \cdot W_C \\ &= \frac{1}{n(n-1)\cdots(n-d+1)(n-d)} \\ &\qquad A_\ell^{d_\ell} \cdots A_{j+1}^{d_{j+1}} A_j^{d_j+1} A_{j-1}^{d_{j-1}} \cdots A_1^{d_1} \cdot W_C. \end{aligned}$$

The converse implication follows from the proof of Theorem 4.2. □

5. Designs and Molien series

Bonnecaze et al. [4] studied certain length of Type II codes to focus some relation between Jacobi polynomials and designs. In this section, we follow the idea, and establish the connection between Jacobi polynomials and designs for some Type III and Type IV codes. We would like to mention that in this section, we study Jacobi polynomials with one reference vector. To overcome all sorts of confusions, we refer the readers to [4] for notations and symbols.

First, let us recall [4] for the definitions of various types of designs. A *design* with parameters t-$(v, k, (\lambda_1^{a_1}, \ldots, \lambda_N^{a_N}))$ is a collection of k-element subsets called *blocks* of a v-element set (the *varieties*) and a partition of the set of all t-tuples into N *groups* such that every t-set belonging to the i^{th} group (comprising a_i such t-sets) is contained in exactly λ_i blocks. Notice that for $N = 1$ the design coincide with a t-design. A *packing* (resp. *covering*) *design* with parameters t-(v, k, λ) is a design with $\max_i(\lambda_i) = \lambda$ (resp. $\min_i(\lambda_i) = \lambda$). The maximum (resp. minimum) number of blocks of a packing (resp. covering) design denoted by $D_\lambda(v, k, t)$ (resp. $C_\lambda(v, k, t)$).

The study of weight polynomials of a code with the help of invariant theory is a very convenient and powerful technique, as shown in [25, 28]. Let G be a finite subgroup of $GL(2, \mathbb{C})$, and G acts on a polynomial ring of two variables, say x, y. Then it is well-known from [28, Theorem 1] that the classical Molien series gives the linearly independent homogeneous invariant polynomials of G. Later, R.P. Stanley [29] introduce the notion of *bivariate Molien series* that computes invariant polynomials of G by their homogeneous degrees in w, z and x, y. R.P. Stanley [29] defined the bivariate Molien series as follows:

$$f(u,v) := \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1-ug)\det(1-vg)}.$$

A. Bonnecaze et al. [4] showed that the bivariate Molien series plays an important role to describe the relation between the Jacobi polynomials of codes and designs such as group divisible designs, packing (resp. covering) designs. To compute all the invariant polynomials of G explicitly, it is convenient to classify the invariants by their degrees. We denote the homogeneous part of degree d of $f(u,v)$ by $f[d]$.

In the following examples, we study two types of codes over $\mathbb{F}_q$; Type III and Type IV, that hold t-designs with parameters t-(v,k,λ), and we would like to give an upper (resp. lower) bound of $D_\lambda(v,k,t)$ (resp. $C_\lambda(v,k,t)$) of a packing (resp. covering) design corresponding to the parameters. To do so, firstly, we compute the homogeneous part $f[d]$ of $f(u,v)$ corresponding to a code of length d. The coefficients of $f[d]$ determine the number of polynomials that are needed to generate the space of Jacobi polynomials corresponding to the reference sets with a particular cardinality. The number of those Jacobi polynomials determines the number of λ's of t-$(n,k,\lambda_1^{a_1},\dots,\lambda_N^{a_N})$. Finally, the coefficient of the term $x^{n-k}y^k$ in the weight enumerator of the code obtains the upper (resp. lower) bound of $D_\lambda(v,k,t)$ (resp. $C_\lambda(v,k,t)$). Note that a packing (resp. covering) design is a simple design.

5.1. **Type III codes.** The MacWilliams identity and the modulo 3 congruence condition yield that the weight enumerator of a Type III code remains invariant under the action of group G_3 of order 48 which is generated by the following two matrices:

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix} \text{ and } \begin{bmatrix}1 & 0\\ 0 & e^{2\pi i/3}\end{bmatrix}.$$

For the case of the group G_3, we get from the Magma computations that the denominator of $f(u,v)$ is in the form of $d(u)d(v)$, where

$$d(u) = (u-1)^2(u+1)^2(u^2+1)^2(u^2-u+1)(u^2+u+1)(u^4-u^2+1).$$

Example 5.1 (length 4)**.** Let C_4^{III} be a ternary self-dual code of length 4 in [18]. Then

$$f[4] = u^4 + u^3v + u^2v^2 + uv^3 + v^4.$$

Since C_4^{III} holds 3-design, we assume that $|T| = 1, 2, 3$. Then

$$\begin{aligned} J_{C_4^{\mathrm{III}},1} &= \frac{1}{4} AW_{C_4^{\mathrm{III}}}(x,y) \\ &= w(x^3 + 2y^3) + 6zxy^2, \\ J_{C_4^{\mathrm{III}},2} &= \frac{1}{4\cdot 3} A^2W_{C_4^{\mathrm{III}}}(x,y) \\ &= w^2x^2 + 4wzy^2 + 4z^2xy, \\ J_{C_4^{\mathrm{III}},3} &= \frac{1}{4\cdot 3\cdot 2} A^3W_{C_4^{\mathrm{III}}}(x,y) \\ &= w^3x + 6wz^2y + 2z^3x. \end{aligned}$$

By dividing the coefficient of the term z^ty^{3-t} $(t = 1, 2, 3)$ in the Jacobi polynomial by 2, we obtain the values of λ. Since the coefficient of the term $u^{4-t}v^t$ in $f[d]$ is 1, we obtain the group divisible design t-$(4, 3, \lambda)$. Then the maximum number of blocks of t-$(4, 3, \lambda)$ design is 4, and the minimum number of blocks of t-$(4, 3, \lambda)$ design is 4. Therefore, $D_\lambda(4, 3, t) \le 4 \le C_\lambda(4, 3, t)$.

Example 5.2 (length 8)**.** Let C_8^{III} be a ternary self-dual code of length 8 in [18]. Then

$$f[8] = u^8 + u^7v + 2u^6v^2 + 2u^5v^3 + 2u^4v^4 + 2u^3v^5 + 2u^2v^6 + uv^7 + v^8.$$

Since C_8^{III} holds 1-design, we assume that $|T| = 1$. Then

$$J_{C_8^{\mathrm{III}},1} = \frac{1}{8} AW_{C_8^{\mathrm{III}}}(x,y) = w(x^7 + 10x^4y^3 + 16xy^6) + z(6x^5y^2 + 48x^2y^5).$$

The space of Jacobi polynomials $J_{C_8^{\mathrm{III}},T}$ with $|T| = 2$ may be generated by the two polynomials

$$\begin{aligned} J^1_{C_8^{\mathrm{III}},2} &= w^2(x^6 + 8x^3y^3) + wz(4x^4y^2 + 32xy^5) + z^2(4x^5y + 32x^2y^4), \\ J^2_{C_8^{\mathrm{III}},2} &= w^2(4x^3y^3 + 4y^6) + wz(12x^4y^2 + 24xy^5) + 36z^2x^2y^4. \end{aligned}$$

Combining these two equations, we obtain 2-designs with parameters

$$\begin{aligned} &2\text{-}(8, 3, (2^{12}, 0^{16})), \\ &2\text{-}(8, 6, (8^{12}, 9^{16})). \end{aligned}$$

Since $k = 3\ell$ $(1 \le \ell \le 2)$, dividing the coefficient of the term z^ty^{k-t} $(t = 2, 3)$ in the Jacobi polynomials by 2^ℓ, we obtain the values of λ_1, λ_2. Dividing the coefficient of the term $x^{8-k}y^k$ in the weight enumerator of the code by 2^ℓ we obtain an upper (resp. lower) bound of $D_\lambda(8,k,t)$ (resp. $C_\lambda(8,k,t)$).

$$D_2(8,3,2) \le 8 \le C_0(8,3,2),$$
$$D_9(8,6,2) \le 16 \le C_8(8,6,2).$$

The space of Jacobi polynomials $J_{C_8^{\rm III},T}$ with $|T|=3$ may be generated by the two polynomials

$$\begin{aligned}
J^1_{C_8^{\rm III},3} &= w^3(x^5+8x^2y^3) + wz^2(6x^4y+48xy^4) + z^3(2x^5+16x^2y^3),\\
J^2_{C_8^{\rm III},3} &= w^3(x^5+2x^2y^3) + w^2z(10x^3y^2+8y^5)\\
&\quad + wz^2(4x^4y+32xy^4) + 24z^3x^2y^3,
\end{aligned}$$

which gives packing and covering designs

$$D_1(8,3,3) \le 8 \le C_0(8,3,3),$$
$$D_6(8,6,3) \le 16 \le C_4(8,6,3).$$

Example 5.3 (length 12)**.** Let $C_{12}^{\rm III}$ be the first ternary self-dual code of length 12 in [18].

$$f[12] = 2u^{12} + 2u^{11}v + 3u^{10}v^2 + 4u^9v^3 + 4u^8v^4 + 4u^7v^5 + 5u^6v^6 + \cdots .$$

Since C_{12}^{III} holds 5-design, we observe that

$$
\begin{aligned}
J_{C_{12}^{\mathrm{III}},1} &= \frac{1}{12} A W_{C_{12}^{\mathrm{III}}}(x,y) \\
&= w(x^{11} + 132x^5y^6 + 110x^2y^9) + z(132x^6y^5 + 330x^3y^8 + 24y^{11}), \\
J_{C_{12}^{\mathrm{III}},2} &= \frac{1}{12\cdot 11} A^2 W_{C_{12}^{\mathrm{III}}}(x,y) \\
&= w^2(x^{10} + 60x^4y^6 + 20xy^9) + 2wz(72x^5y^5 + 90x^2y^8) \\
&\quad + z^2(60x^6y^4 + 240x^3y^7 + 24y^{10}), \\
J_{C_{12}^{\mathrm{III}},3} &= \frac{1}{12\cdot 11\cdot 10} A^3 W_{C_{12}^{\mathrm{III}}}(x,y) \\
&= w^3(x^9 + 24x^3y^6 + 2y^9) + w^2z(108x^4y^5 + 54xy^8) \\
&\quad + wz^2(108x^5y^4 + 216x^2y^7) \\
&\quad + z^3(24x^6y^3 + 168x^3y^6 + 24y^9), \\
J_{C_{12}^{\mathrm{III}},4} &= \frac{1}{12\cdot 11\cdot 10\cdot 9} A^4 W_{C_{12}^{\mathrm{III}}}(x,y) \\
&= w^4(x^8 + 8x^2y^6) + w^3z(64x^3y^5 + 8y^8) + w^2z^2(120x^4y^4 + 96xy^7) \\
&\quad + wz^3(64x^5y^3 + 224x^2y^6) + z^4(8x^6y^2 + 112x^3y^5 + 24y^8), \\
J_{C_{12}^{\mathrm{III}},5} &= \frac{1}{12\cdot 11\cdot 10\cdot 9\cdot 8} A^5 w_{C_{12}^{\mathrm{III}}}(x,y) \\
&= w^5(x^7 + 2xy^6) + 30w^4zx^2y^5 + w^3z^2(100x^3y^4 + 20y^7) \\
&\quad + wz^4(30x^5y^2 + 210x^2y^5) + w^2z^3(100x^4y^3 + 140xy^6) \\
&\quad + z^5(2x^6y + 70x^3y^4 + 24y^7).
\end{aligned}
$$

The space of Jacobi polynomials $J_{C_{12}^{\mathrm{III}},T}$ with $|T| = 6$ is generated by the two polynomials

$$
\begin{aligned}
J^1_{C_{12}^{\mathrm{III}},6} &= w^6(x^6 + 2y^6) + 90w^4z^2x^2y^4 + w^3z^3(80x^3y^3 + 40y^6) \\
&\quad + w^2z^4(90x^4y^2 + 180xy^5) + 180wz^5x^2y^4 \\
&\quad + z^6(2x^6 + 40x^3y^3 + 24y^6), \\
J^2_{C_{12}^{\mathrm{III}},6} &= w^6x^6 + 12w^5zxy^5 + 60w^4z^2x^2y^4 + w^3z^3(120x^3y^3 + 40y^6) \\
&\quad + w^2z^4(60x^4y^2 + 180xy^5) + wz^5(12x^5y + 180x^2y^4) \\
&\quad + z^6(40x^3y^3 + 24y^6),
\end{aligned}
$$

which gives packing and covering designs

$$D_1(12,6,6) \le 132 \le C_0(12,6,6).$$

Example 5.4 (length 16). Let C_{16}^{III} be the seventh ternary self-dual code of length 16 in [18].

$$f[16] = 2u^{16} + 3u^{15}v + 4u^{14}v^2 + 5u^{13}v^3 + 6u^{12}v^4 + 6u^{11}v^5 + 7u^{10}v^6 + 7u^9v^7 + 7u^8v^8 + \cdots .$$

Observe that

$$\begin{aligned}
J_{C_{16}^{\mathrm{III}},1} &= \frac{1}{16} A W_{C_{16}^{\mathrm{III}}}(x,y) \\
&= w(x^{15} + 140x^9y^6 + 1190x^6y^9 + 840x^3y^{12} + 16y^{15}) \\
&\quad + z(84x^{10}y^5 + 1530x^7y^8 + 2520x^4y^{11} + 240xy^{14}), \\
J_{C_{16}^{\mathrm{III}},2} &= \frac{1}{16\cdot 15} A^2 W_{C_{16}^{\mathrm{III}}}(x,y) \\
&= w^2(x^{14} + 84x^8y^6 + 476x^5y^9 + 168x^2y^{12}) \\
&\quad + wz(112x^9y^5 + 1428x^6y^8 + 1344x^3y^{11} + 32y^{14}) \\
&\quad + z^2(28x^{10}y^4 + 816x^7y^7 + 1848x^4y^{10} + 224xy^{13}), \\
J_{C_{16}^{\mathrm{III}},3} &= \frac{1}{16\cdot 15\cdot 14} A^3 W_{C_{16}^{\mathrm{III}}}(x,y) \\
&= w^3(x^{13} + 48x^7y^6 + 170x^4y^9 + 24xy^{12}) \\
&\quad + w^2z(108x^8y^5 + 918x^5y^8 + 432x^2y^{11}) \\
&\quad + wz^2(60x^9y^4 + 1224x^6y^7 + 1584x^3y^{10} + 48y^{13}) \\
&\quad + z^3(8x^{10}y^3 + 408x^7y^6 + 1320x^4y^9 + 208xy^{12}).
\end{aligned}$$

The space of Jacobi polynomials $J_{C_{16}^{\mathrm{III}},T}$ with $|T| = 4$ may be generated by the four polynomials

$$\begin{aligned}
J^1_{C_{16}^{\mathrm{III}},4} &= w^4(x^{12} + 32x^6y^6 + 40x^3y^9 + 8y^{12}) \\
&\quad + w^3z(64x^7y^5 + 520x^4y^8 + 64xy^{11}) \\
&\quad + w^2z^2(120x^8y^4 + 1056x^5y^7 + 768x^2y^{10}) \\
&\quad + wz^3(928x^6y^6 + 1600x^3y^9 + 64y^{12}) \\
&\quad + z^4(8x^{10}y^2 + 176x^7y^5 + 920x^4y^8 + 192xy^{11}), \\
J^2_{C_{16}^{\mathrm{III}},4} &= w^4(x^{12} + 24x^6y^6 + 56x^3y^9) \\
&\quad + w^3z(96x^7y^5 + 456x^4y^8 + 96xy^{11}) \\
&\quad + w^2z^2(72x^8y^4 + 1152x^5y^7 + 720x^2y^{10}) \\
&\quad + wz^3(32x^9y^3 + 864x^6y^6 + 1632x^3y^9 + 64y^{12}) \\
&\quad + z^4(192x^7y^5 + 912x^4y^8 + 192xy^{11}), \\
J^3_{C_{16}^{\mathrm{III}},4} &= w^4(x^{12} + 26x^6y^6 + 52x^3y^9 + 2y^{12}) \\
&\quad + w^3z(88x^7y^5 + 472x^4y^8 + 88xy^{11}) \\
&\quad + w^2z^2(84x^8y^4 + 1128x^5y^7 + 732x^2y^{10}) \\
&\quad + wz^3(24x^9y^3 + 880x^6y^6 + 1624x^3y^9 + 64y^{12}) \\
&\quad + z^4(2x^{10}y^2 + 188x^7y^5 + 914x^4y^8 + 192xy^{11}), \\
J^4_{C_{16}^{\mathrm{III}},4} &= w^4(x^{12} + 28x^6y^6 + 48x^3y^9 + 4y^{12}) \\
&\quad + w^3z(80x^7y^5 + 488x^4y^8 + 80xy^{11}) \\
&\quad + w^2z^2(72x^8y^4 + 1104x^5y^7 + 744x^2y^{10}) \\
&\quad + wz^3(16x^9y^3 + 896x^6y^6 + 1616x^3y^9 + 64y^{12}) \\
&\quad + z^4(4x^{10}y^2 + 184x^7y^5 + 916x^4y^8 + 192xy^{11}),
\end{aligned}$$

which gives packing and covering designs

$$\begin{aligned}
D_4(16,6,4) \le 112 \le C_0(16,6,4), \\
D_{96}(16,9,4) \le 1360 \le C_{88}(16,9,4), \\
D_{343}(16,12,4) \le 1260 \le C_{342}(16,12,4).
\end{aligned}$$

Example 5.5 (length 20)**.** Let C_{20}^{III} be the 19th ternary self-dual code of length 20 in [18].

$$\begin{aligned}
f[20] = 2u^{20} + 3u^{19}v + 5u^{18}v^2 + 6u^{17}v^3 + 7u^{16}v^4 + 8u^{15}v^5 + 9u^{14}v^6 \\
+ 9u^{13}v^7 + 10u^{12}v^8 + 10u^{11}v^9 + 10u^{10}v^{10} + \cdots .
\end{aligned}$$

Observe that

$$\begin{aligned} J_{C_{20}^{\mathrm{III}},1} &= \frac{1}{20} AW_{C_{20}^{\mathrm{III}}}(x,y) \\ &= w(x^{19} + 84x^{13}y^6 + 2398x^{10}y^9 + 10512x^7y^{12} + 6432x^4y^{15} + 256xy^{18}) \\ &\quad + z(36x^{14}y^5 + 1962x^{11}y^8 + 15768x^8y^{11} + 19296x^5y^{14} + 2304x^2y^{17}). \end{aligned}$$

The space of Jacobi polynomials $J_{C_{20}^{\mathrm{III}},T}$ with $|T| = 2$ may be generated by the two polynomials

$$\begin{aligned} J^1_{C_{20}^{\mathrm{III}},2} &= w^2(x^{18} + 48x^{12}y^6 + 1300x^9y^9 + 3816x^6y^{12} + 1392x^3y^{15} + 4y^{18}) \\ &\quad + wz(72x^{13}y^5 + 2196x^{10}y^8 + 13392x^7y^{11} + 10080x^4y^{14} + 504xy^{17}) \\ &\quad + z^2(864x^{11}y^7 + 9072x^8y^{10} + 14256x^5y^{13} + 2052x^2y^{16}), \\ J^2_{C_{20}^{\mathrm{III}},2} &= w^2(x^{18} + 68x^{12}y^6 + 1220x^9y^9 + 3936x^6y^{12} + 1312x^3y^{15} + 24y^{18}) \\ &\quad + wz(32x^{13}y^5 + 2356x^{10}y^8 + 13152x^7y^{11} + 10240x^4y^{14} + 464xy^{17}) \\ &\quad + z^2(20x^{14}y^4 + 784x^{11}y^7 + 9192x^8y^{10} + 14176x^5y^{13} + 2072x^2y^{16}), \end{aligned}$$

which gives packing and covering designs

$$\begin{aligned} D_{10}(20,6,2) &\leq 60 \leq C_0(20,6,2), \\ D_{432}(20,9,2) &\leq 2180 \leq C_{392}(20,9,2), \\ D_{4296}(20,12,2) &\leq 12240 \leq C_{4212}(20,12,2), \\ D_{6444}(20,15,2) &\leq 11544 \leq C_{6308}(20,15,2). \end{aligned}$$

5.2. **Type IV codes.** It is well-known (see [25]) that the weight enumerator of a Type IV code remains invariant under the action of group G_4 of order 12 which is generated by the following two matrices:

$$\frac{1}{2}\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which corresponds to the MacWilliams identity and the modulo 2 congruence condition, respectively. In particular, for the case of the group G_4 a Magma computation gives the denominator $d(u)d(v)$ of $f(u,v)$, where

$$d(u) = (1-u+u^2)(1+u+u^2)(1+2u^6+3u^{12}+4u^{18}+5u^{24}+6u^{30}+7u^{36}).$$

Example 5.6 (length 2)**.** Let C_2^{IV} be a Hermitian self-dual code over $\mathbb{F}_4$ of length 2 in [18]. Then

$$f[2] = u^2 + uv + v^2$$

If $|T| = 1$, we have

$$\begin{aligned} J_{C_2^{\mathrm{IV}},1} &= \frac{1}{2} AW_{C_2^{\mathrm{IV}}}(x,y) \\ &= wx + 3zy. \end{aligned}$$

Example 5.7 (length 4)**.** Let C_4^{IV} be a Hermitian self-dual code over $\mathbb{F}_4$ of length 4 in [18].

$$f[4] = u^4 + u^3v + 2u^2v^2 + uv^3 + v^4.$$

Observe that

$$J_{C_4^{\mathrm{IV}},1} = \frac{1}{4} AW_{C_4^{\mathrm{IV}}}(x,y) = wx^3 + 3wxy^2 + 3zx^2y + 9zy^3.$$

The space of Jacobi polynomials $J_{C_4^{\mathrm{IV}},T}$ with $|T| = 2$ is generated by the two polynomials

$$\begin{aligned} J^1_{C_4^{\mathrm{IV}},2} &= w^2x^2 + 6wzxy + 9z^2y^2, \\ J^2_{C_4^{\mathrm{IV}},2} &= w^2x^2 + 3w^2y^2 + 3z^2x^2 + 9z^2y^2. \end{aligned}$$

Combining these two equations we obtain 2-designs with parameters

$$2\text{-}(4, 2, 0^4, 1^2).$$

Since $k = 2$, dividing the coefficient of the term z^4y^2 in the Jacobi polynomials by 2, we obtain the values of λ_1, λ_2. This gives packing and covering designs

$$D_1(4, 2, 2) \leq 2 \leq C_0(4, 2, 2).$$

Example 5.8 (length 6)**.** Let C_6^{IV} be the first Hermitian self-dual code over $\mathbb{F}_4$ of length 6 in [18].

$$f[6] = 2u^6 + 2u^5v + 3u^4v^2 + 3u^3v^3 + 3u^2v^4 + 2uv^5 + 2v^6.$$

Observe that

$$J_{C_6^{\mathrm{IV}},1} = \frac{1}{6} AW_{C_6^{\mathrm{IV}}}(x,y) = wx^5 + 6wx^3y^2 + 9wxy^4 + 3zx^4y + 18zx^2y^3 + 27y^5.$$

The space of Jacobi polynomials $J_{C_6^{\mathrm{IV}},T}$ with $|T| = 2$ may be generated by the two polynomials

$$\begin{aligned} J^1_{C_6^{\mathrm{IV}},2} &= w^2(x^4 + 6x^2y^2 + 9y^4) + z^2(3x^4 + 18x^2y^2 + 27y^4), \\ J^2_{C_6^{\mathrm{IV}},2} &= w^2(x^4 + 3x^2y^2) + wz(6x^3y + 18xy^3) + z^2(9x^2y^2 + 27y^4). \end{aligned}$$

Combining these two equations we obtain 2-designs with parameters

$$\begin{aligned} &2\text{-}(6, 2, 0^{12}, 1^3), \\ &2\text{-}(6, 4, 1^{12}, 2^3). \end{aligned}$$

Since $k = 2\ell$ $(1 \le \ell \le 2)$, by dividing the coefficient of the term z^2y^{k-2} in the Jacobi polynomials by 3^ℓ, we obtain the values of λ_1, λ_2. This gives packing and covering designs

$$D_1(6,2,2) \le 3 \le C_0(6,2,2),$$
$$D_2(6,4,2) \le 3 \le C_1(6,4,2).$$

Example 5.9 (length 8)**.** Let C_8^{IV} be the third Hermitian self-dual code over $\mathbb{F}_4$ of length 8 in [18].

$$f[8] = 2u^8 + 3u^7v + 4u^6v^2 + 4u^5v^3 + 5u^4v^4 + 4u^3v^5 + 4u^2v^6 + 3uv^7 + 2v^8.$$

Observe that

$$\begin{aligned}
J_{C_8^{\mathrm{IV}},1} &= \frac{1}{8} AW_{C_8^{\mathrm{IV}}}(x,y)\\
&= w(x^7 + 21x^3y^4 + 42xy^6) + z(21x^4y^3 + 126x^2y^5 + 45y^7),\\
J_{C_8^{\mathrm{IV}},2} &= \frac{1}{8\cdot 7} A^2W_{C_8^{\mathrm{IV}}}(x,y)\\
&= w^2(x^6 + 9x^2y^4 + 6y^6) + wz(24x^3y^3 + 72xy^5)\\
&\quad + z^2(9x^4y^2 + 90x^2y^4 + 45y^6),\\
J_{C_8^{\mathrm{IV}},3} &= \frac{1}{8\cdot 7\cdot 6} A^3W_{C_8^{\mathrm{IV}}}(x,y)\\
&= w^3(x^5 + 3xy^4) + w^2z(18x^2y^3 + 18y^5)\\
&\quad + wz^2(18x^3y^2 + 90xy^4) + z^3(3x^4y + 60x^2y^3 + 45y^5).
\end{aligned}$$

The space of Jacobi polynomials $J_{C_8^{\mathrm{IV}},T}$ with $|T| = 4$ may be generated by the two polynomials

$$\begin{aligned}
J^1_{C_8^{\mathrm{IV}},4} &= w^4(x^4 + 3y^4) + 36w^2z^2(x^2y^2 + 36y^4)\\
&\quad + 96wz^3xy^3 + z^4(3x^4 + 36x^2y^2 + 45y^4),\\
J^2_{C_8^{\mathrm{IV}},4} &= w^4x^4 + 12w^3zxy^3 + w^2z^2(18x^2y^2 + 36y^4)\\
&\quad + wz^3(12x^3y + 96xy^3) + z^4(36x^2y^2 + 45y^4).
\end{aligned}$$

Combining these two equations we obtain 4-designs with parameters

$$4\text{-}(8, 4, 0^{56}, 1^{14}).$$

This gives packing and covering designs

$$D_1(8,4,4) \le 14 \le C_0(8,4,4).$$

Example 5.10 (length 12)**.** Let C_{12}^{IV} be the seventh Hermitian self-dual code over $\mathbb{F}_4$ of length 12 in [18].

$$f[12] = 3u^{12} + 4u^1 1v + 6u^{10}v^2 + 7u^9v^3 + 8u^8v^4 + 8u^7v^5 + 9u^6v^6 + 8u^5v^7 + 8u^4v^8 + 7u^3v^9 + 6u^2v^{10} + 4uv^{11} + 3v^{12}.$$

Observe that

$$\begin{aligned} J_{C_{12}^{\mathrm{IV}},1} &= \frac{1}{12}AW_{C_{12}^{\mathrm{IV}}}(x,y) \\ &= w(x^{11} + 30x^7y^4 + 108x^5y^6 + 585x^3y^8 + 300xy^{10}) \\ &\quad + z(15x^8y^3 + 108x^6y^5 + 1170x^4y^7 + 279y^{11}). \end{aligned}$$

The space of Jacobi polynomials $J_{C_{12}^{\mathrm{IV}},T}$ with $|T| = 2$ may be generated by the two polynomials

$$\begin{aligned} J^1_{C_{12}^{\mathrm{IV}},2} &= w^2(x^{10} + 30x^6y^4 + 60x^4y^6 + 105x^2y^8 + 60y^{10}) \\ &\quad + wz(96x^5y^5 + 960x^3y^7 + 480xy^9) \\ &\quad + z^2(15x^8y^2 + 60x^6y^4 + 690x^4y^6 + 1260x^2y^8 + 279y^{10}), \\ J^2_{C_{12}^{\mathrm{IV}},2} &= w^2(x^{10} + 18x^6y^4 + 48x^4y^6 + 165x^2y^8 + 24y^{10}) \\ &\quad + wz(24x^7y^3 + 120x^5y^5 + 840x^3y^7 + 552xy^9) \\ &\quad + z^2(3x^8y^2 + 48x^6y^4 + 750x^4y^6 + 1224x^2y^8 + 279y^{10}), \end{aligned}$$

which gives packing and covering designs

$$\begin{aligned} D_5(12,4,2) &\le 15 \le C_1(12,4,2), \\ D_{12}(12,6,2) &\le 52 \le C_{10}(12,6,2), \\ D_{110}(12,8,2) &\le 255 \le C_{90}(12,8,2). \end{aligned}$$

Example 5.11 (length 14)**.** Let C_{14}^{IV} be the first Hermitian self-dual code over $\mathbb{F}_4$ of length 14 in [18].

$$\begin{aligned} f[14] &= 3u^{14} + 5u^{13}v + 7u^{12}v^2 + 8u^{11}v^3 + 10u^{10}v^4 + 10u^9v^5 + 11u^8v^6 \\ &\quad + 11u^7v^7 + 11u^6v^8 + 10u^5v^9 + 10u^4v^{10} + 8u^3v^{11} + 7u^2v^{12} \\ &\quad + 5uv^{13} + 3v^{14}. \end{aligned}$$

Observe that

$$\begin{aligned} J_{C_{14}^{\mathrm{IV}},1} &= \frac{1}{14}AW_{C_{14}^{\mathrm{IV}}}(x,y) \\ &= w(x^{13} + 18x^{11}y^2 + 135x^9y^4 + 540x^7y^6 + 1215x^5y^8 + 1458x^3y^{10} \\ &\quad + 729xy^{12}) + z(3x^{12}y + 54x^{10}y^3 + 405x^8y^5 + 1620x^6y^7 + 3645x^4y^9 \\ &\quad + 4374x^2y^{11} + 2187y^{13}). \end{aligned}$$

The space of Jacobi polynomials $J_{C_{14}^{\rm IV},T}$ with $|T|=2$ may be generated by the two polynomials

$$\begin{aligned}J^1_{C_{14}^{\rm IV},2} &= w^2(x^{12}+18x^{10}y^2+135x^8y^4+540x^6y^6+1215x^4y^8+1458x^2y^{10}\\ &\quad +729y^{12})+z^2(3x^{12}+54x^{10}y^2+405x^8y^4+1620x^6y^6+3645x^4y^8\\ &\quad +4374x^2y^{10}+2187y^{12}),\\ J^2_{C_{14}^{\rm IV},2} &= w^2(x^{12}+15x^{10}y^2+90x^8y^4+270x^6y^6+405x^4y^8+243x^2y^{10})\\ &\quad +wz(6x^{11}y+90x^9y^3+540x^7y^5+1620x^5y^7+2430x^3y^9\\ &\quad +1458xy^{11})+z^2(9x^{10}y^2+135x^8y^4+810x^6y^6+2430x^4y^8\\ &\quad +3645x^2y^{10}+2187y^{12}),\end{aligned}$$

which gives packing and covering designs

$$\begin{aligned}D_1(14,2,2)&\le 7\le C_0(14,2,2),\\ D_6(14,4,2)&\le 21\le C_1(14,4,2),\\ D_{15}(14,6,2)&\le 35\le C_5(14,6,2),\\ D_{20}(14,8,2)&\le 35\le C_{10}(14,8,2),\\ D_{15}(14,10,2)&\le 21\le C_{10}(14,10,2),\\ D_6(14,12,2)&\le 7\le C_5(14,12,2).\end{aligned}$$

Example 5.12 (length 16)**.** Let $C_{16}^{\rm IV}$ be the 35th Hermitian self-dual code over $\mathbb{F}_4$ of length 16 in [18].

$$\begin{aligned}f[16] &= 3u^{16}+5u^{15}v+8u^{14}v^2+9u^{13}v^3+11u^{12}v^4+12u^{11}v^5+13u^{10}v^6\\ &\quad +13u^9v^7+14u^8v^8+13u^7v^9+13u^6v^{10}+12u^5v^{11}+11u^4v^{12}\\ &\quad +9u^3v^{13}+8u^2v^{14}+5uv^{15}+3v^{16}.\end{aligned}$$

Observe that

$$\begin{aligned}J_{C_{16}^{\rm IV},1} &= \frac{1}{16}AW_{C_{16}^{\rm IV}}(x,y)\\ &= w(x^{15}+21x^{13}y^2+189x^{11}y^4+945x^9y^6+2835x^7y^8+5103x^5y^{10}\\ &\quad +5103x^3y^{12}+2187xy^{14})+z(3x^{14}y+63x^{12}y^3+567x^{10}y^5\\ &\quad +2835x^8y^7+8505x^6y^9+15309x^4y^{11}+15309x^2y^{13}+6561y^{15}).\end{aligned}$$

The space of Jacobi polynomials $J_{C^{\mathrm{IV}}_{16},T}$ with $|T| = 2$ is generated by the following two polynomials:

$$\begin{aligned}
J^1_{C^{\mathrm{IV}}_{16},2} &= w^2(x^{14} + 21x^{12}y^2 + 189x^{10}y^4 + 945x^8y^6 + 2835x^6y^8 + 5103x^4y^{10} \\
&\quad + 5103x^2y^{12} + 2187y^{14}) + z^2(3x^{14} + 63x^{12}y^2 + 567x^{10}y^4 \\
&\quad + 2835x^8y^6 + 8505x^6y^8 + 15309x^4y^{10} + 15309x^2y^{12} + 6561y^{14}), \\
J^2_{C^{\mathrm{IV}}_{16},2} &= w^2(x^{14} + 18x^{12}y^2 + 135x^{10}y^4 + 540x^8y^6 + 1215x^6y^8 + 1458x^4y^{10} \\
&\quad + 729x^2y^{12}) + wz(6x^{13}y + 108x^{11}y^3 + 810x^9y^5 + 3240x^7y^7 \\
&\quad + 7290x^5y^9 + 8748x^3y^{11} + 4374xy^{13}) + z^2(9x^{12}y^2 + 162x^{10}y^4 \\
&\quad + 1215x^8y^6 + 4860x^6y^8 + 10935x^4y^{10} + 13122x^2y^{12} + 6561y^{14}),
\end{aligned}$$

which gives packing and covering designs

$$\begin{aligned}
D_1(16,2,2) &\le 8 \le C_0(16,2,2), \\
D_7(16,4,2) &\le 28 \le C_1(16,4,2), \\
D_{21}(16,6,2) &\le 56 \le C_6(16,6,2), \\
D_{35}(16,8,2) &\le 70 \le C_{15}(16,8,2), \\
D_{35}(16,10,2) &\le 56 \le C_{20}(16,10,2), \\
D_{21}(16,12,2) &\le 28 \le C_{15}(16,12,2), \\
D_7(16,14,2) &\le 8 \le C_6(16,14,2).
\end{aligned}$$

Example 5.13 (length 18)**.** Let C^{IV}_{18} be the 225th Hermitian self-dual code over $\mathbb{F}_4$ of length 18 in [18].

$$\begin{aligned}
f[18] &= 4u^{18} + 6u^{17}v + 9u^{16}v^2 + 11u^{15}v^3 + 13u^{14}v^4 + 14u^{13}v^5 + 16u^{12}v^6 \\
&\quad + 16u^{11}v^7 + 17u^{10}v^8 + 17u^9v^9 + 17u^8v^{10} + 16u^7v^{11} + 16u^6v^{12} \\
&\quad + 14u^5v^{13} + 13u^4v^{14} + 11u^3v^{15} + 9u^2v^{16} + 6uv^{17} + 4v^{18}.
\end{aligned}$$

Observe that

$$\begin{aligned}
J_{C^{\mathrm{IV}}_{18},1} &= \frac{1}{18} AW_{C^{\mathrm{IV}}_{18}}(x,y) \\
&= w(x^{17} + 24x^{15}y^2 + 252x^{13}y^4 + 1512x^{11}y^6 + 5670x^9y^8 \\
&\quad + 13608x^7y^{10} + 20412x^5y^{12} + 17496x^3y^{14} + 6561xy^{16}) \\
&\quad + z(3x^{16}y + 72x^{14}y^3 + 756x^{12}y^5 + 4536x^{10}y^7 + 17010x^8y^9 \\
&\quad + 40824x^6y^{11} + 61236x^4y^{13} + 52488x^2y^{15} + 19683y^{17}).
\end{aligned}$$

The space of Jacobi polynomials $J_{C_{18}^{\mathrm{IV}},T}$ with $|T| = 2$ may be generated by the two polynomials

$$\begin{aligned}
J^1_{C_{18}^{\mathrm{IV}},2} = {} & w^2(24x^{14}y^2 + 252x^{12}y^4 + 1512x^{10}y^6 + 5670x^8y^8 + 13608x^6y^{10} \\
& + 20412x^4y^{12} + 17496x^2y^{14} + 6561y^{16}) + z^2(3x^{16} + 72x^{14}y^2 \\
& + 756x^{12}y^4 + 4536x^{10}y^6 + 17010x^8y^8 + 40824x^6y^{10} \\
& + 61236x^4y^{12} + 52488x^2y^{14}), \\
J^2_{C_{18}^{\mathrm{IV}},2} = {} & w^2(21x^{14}y^2 + 189x^{12}y^4 + 945x^{10}y^6 + 2835x^8y^8 + 5103x^6y^{10} \\
& + 5103x^4y^{12} + 2187x^2y^{14}) + wz(6x^{15}y + 126x^{13}y^3 + 1134x^{11}y^5 \\
& + 5670x^9y^7 + 17010x^7y^9 + x^5y^{11} + 30618x^3y^{13} + 13122xy^{15}) \\
& + z^2(9x^{14}y^2 + 189x^{12}y^4 + 1701x^{10}y^6 + 8505x^8y^8 + 25515x^6y^{10} \\
& + 45927x^4y^{12} + 45927x^2y^{14}),
\end{aligned}$$

which gives packing and covering designs

$$\begin{aligned}
D_1(18, 2, 2) &\le 9 \le C_0(18, 2, 2), \\
D_8(18, 4, 2) &\le 36 \le C_1(18, 4, 2), \\
D_{28}(18, 6, 2) &\le 84 \le C_7(18, 6, 2), \\
D_{56}(18, 8, 2) &\le 126 \le C_{21}(18, 8, 2), \\
D_{70}(18, 10, 2) &\le 126 \le C_{35}(18, 10, 2), \\
D_{56}(18, 12, 2) &\le 84 \le C_{35}(18, 12, 2), \\
D_{28}(18, 14, 2) &\le 36 \le C_{21}(18, 14, 2), \\
D_8(18, 16, 2) &\le 9 \le C_7(18, 16, 2).
\end{aligned}$$

6. Concluding remarks

The g-th Jacobi polynomials of a binary code were introduced in [19] which were generalized in [11] to the case of a non-binary code. This rises a natural question: is there any possibility to give a generalization of Theorem 4.3 for higher genus cases? We shall answer this question in [9]. The study of this paper will be continued in [13] to the case colored t-design, the idea that was introduce in [5]. Moreover, we shall give the generalizations of the results in [11] for the g-th Jacobi polynomial with multiple reference vectors in [12].

Acknowledgements

This work was supported by JSPS KAKENHI (22K03277).

References

[1] E.F. Assmus, Jr., H.F. Mattson, Jr., New 5-designs, *J. Combinatorial Theory* **6** (1969), 122–151.

[2] E. Bannai, M. Ozeki, Construction of Jacobi forms from certain combinatorial polynomials, *Proc. Japan Acad. Ser. A Math. Sci.* **72** (1996), no. 1, 12–15.

[3] E.R. Berlekamp, F.J. MacWilliams, N.J.A. Sloane, Gleason's theorem on self-dual codes, *IEEE Trans. Inform. Theory* **IT-18** (1972), 409–414.

[4] A. Bonnecaze, B. Mourrain, P. Solé, Jacobi polynomials, type II codes, and designs, *Des. Codes Cryptogr.* **16** (1999), no. 3, 215–234.

[5] A. Bonnecaze, E. Rains, P. Solé, 3-colored 5-designs and $\mathbb{Z}_4$-codes, *J. Stat. Plan. Inference* **86** (2000), 349–368.

[6] W. Bosma, J. Cannon, C. Playoust, The Magma algebra system. I. The user language, *J. Symb. Comp.* **24** (1997), 235–265.

[7] P.J. Cameron, A generalisation of t-designs, *Discrete Math.*, **309**(2009), 4835–4842.

[8] J. H. Conway, N.J.A. Sloane, *Sphere Packings Lattices and Groups*, third edition, Springer, New York, 1999.

[9] H.S. Chakraborty, N. Hamid, T. Miezaki, M. Oura, Jacobi polynomials, Invariant rings, and generalized t-designs I, in preparation.

[10] H.S. Chakraborty, T. Miezaki, Variants of Jacobi polynomials in coding theory, *Des. Codes Cryptogr.*, https://doi.org/10.1007/s10623-021-00923-2

[11] H.S. Chakraborty, T. Miezaki, M. Oura, Weight enumerators, intersection enumerators and Jacobi polynomials II, *Discrete Mathematics* **345**(12) (2022), 113098.

[12] H.S. Chakraborty, T. Miezaki, M. Oura, Weight enumerators, intersection enumerators and Jacobi polynomials III, in preparation.

[13] H.S. Chakraborty, R. Ishikawa, T. Miezaki, Y. Tanaka, Jacobi polynomials and design theory II, in preparation.

[14] P. Delsarte, Hahn polynomials, discrete harmonics, and t-designs, *SIAM J. Appl. Math.* **34**(1) (1978), 157-166.

[15] W. Duke, On codes and Siegel modular forms, *Int. Math. Res. Not.* **5** (1993), 125–136.

[16] M. Eichler, D. Zagier, *The Theory of Jacobi Forms*, Progress in Mathematics, vol. 55, Birkhauser Boston, Inc., Boston, MA, 1985.

[17] A.M. Gleason, Weight polynomials of self-dual codes and the MacWilliams identities, in: Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 3, Gauthier-Villars, Paris, 1971, pp. 211–215.

[18] M. Harada and A. Munemasa, Database of self-dual codes, https://www.math.is.tohoku.ac.jp/˜munemasa/selfdualcodes.htm.

[19] K. Honma, T. Okabe, and M. Oura, Weight enumerators, intersection enumerators, and Jacobi polynomials, *Discrete Math.* **343** (2020), no. 6, 111815.

[20] W. C. Huffman, V. Pless, *Fundamentals of Error-Correcting Codes*, first edition, Cambridge University Press, United Kingdom, 2003.

[21] F.J. MacWilliams, C.L. Mallows, N.J.A. Sloane, Generalizations of Gleason's theorem on weight enumerators of self-dual codes, *IEEE Trans. Inform. Theory* **IT-18** (1972) 794–805.

[22] F.J. MacWilliams, N.J.A. Sloane, *The Theory of Error-Correcting Codes*, first edition, Elsevier/North Holland, New York, 1977.

[23] T. Miezaki, Design-theoretic analogies between codes, lattices, and vertex operator algebras, *Des. Codes Cryptogr.*, **89** (2021), 763–780.

[24] T. Miezaki, A. Munemasa and H. Nakasora, An note on Assmus–Mattson theorems, *Des. Codes Cryptogr.*, **89** (2021), 843–858.

[25] G. Nebe, E.M. Rains, N.J.A. Sloane, *Self-Dual Codes and Invariant Theory*, Algorithms and Computation in Mathematics, vol. **17**, Springer-Verlag, Berlin, 2006.

[26] M. Ozeki, On the notion of Jacobi polynomials for codes, *Math. Proc. Cambridge Philos. Soc.* **121** (1) (1997) 15–30.

[27] B. Runge, Codes and Siegel modular forms, *Discrete Math.* **148** (1–3) (1996) 175–204.

[28] N.J.A. Sloane, Error-correcting codes and invariant theory: New applications of a nineteenth-century technique, *Amer. Math. Monthly* **84** (1977), 82–107.

[29] R.P. Stanley, Invariants of finite groups and their applications to combinatorics, *Bull. (New Series) Amer. Math. Soc.* **1**(3) (1979), 475–511.

Department of Mathematics, Shahjalal University of Science and Technology, Sylhet 3114, BANGLADESH
Email address: himadri-mat@sust.edu

School of Fundamental Science and Engineering, Waseda University, Tokyo 169-8555, Japan
Email address: miezaki@waseda.jp

School of Natural Science and Technology, Kanazawa University, Ishikawa 920-1192,Japan
Email address: oura@se.kanazawa-u.ac.jp

Graduate School of Science and Engineering, Waseda University, Tokyo, 169-8555, Japan
Email address: tanaka_yuuho_dc@akane.waseda.jp