OBSERVATION ON THE WEIGHT ENUMERATORS FROM CLASSICAL INVARIANT THEORY

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The purpose of this paper is to present some relationships among invariant rings. This is done by combining two maps, the Broué–Enguehard map and Igusa's ρ homomorphism. For the completeness of the story, some formulae and statements are given which are not necessarily needed in the present manuscript. Sections 1 and 2 have some expository nature and contain no new result.

1. Classical Invariant Theory. In this section we recall classical invariant theory. For the detail we refer to [18]. We consider a homogeneous polynomial

$$\sum_{i=0}^{n}\binom{n}{i}u_i x^{n-i}y^i$$

of degree n in 2 variables $x,\ y$. The group $SL(2,\mathbf{C})$ operates on the variable space and, if we require that the above form is invariant, the same group operates on the coefficient space. In this way, we get an irreducible representation of $SL(2,\mathbf{C})$ of degree $n+1$. We consider the graded ring of polynomials in the $u_0,u_1,\dots,u_n$ with coefficients in $\mathbf{C}$ and operate $SL(2,\mathbf{C})$ on this graded ring using its action on its homogeneous part of degree one defined by the above representation. Then, the invariant subring $S(2,n)$ is a graded, integrally closed, integral domain over $\mathbf{C}$. In the present paper we deal with $S(2,4)$ and $S(2,6)$. The structure theorems of those rings are established in the 19th century and we shall describe them. The invariant ring $S(2,4)$ is generated by $P,\ Q$, which are algebraically independent. The explicit forms are

$$P = u_0u_4 - 4u_1u_3 + 3u_2^2,$$
$$Q = \det\begin{pmatrix} u_0 & u_1 & u_2\\ u_1 & u_2 & u_3\\ u_2 & u_3 & u_4\end{pmatrix}$$

and the dimension formula of $S(2,4)$ is

$$\frac{1}{(1-t^2)(1-t^3)} = 1+t^2+t^3+t^4+t^5+2t^6+t^7+2t^8+2t^9+2t^{10}+2t^{11}+3t^{12}+\cdots$$

in which the coefficient of t^k denotes the dimension of degree k-part of $S(2,4)$. The invariant ring $S(2,6)$ is generated by $J_2,J_4,J_6,J_{10},J_{15}$. We give the definitions of $J_2,\dots,J_{15}$ in the appendix, taken from [18]. Among them J_2,J_4,J_6,J_{10} are algebraically independent. The ring $\mathbf{C}[J_2,J_4,J_6,J_{10}]$ contains J_{15}^2 but not J_{15}. We also give the explicit formula for J_{15}^2 in the appendix. The dimension formula of $S(2,6)$ is given by

$$\begin{aligned}\frac{1+t^{15}}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})} &= 1+t^2+2\,t^4+3t^6+4t^8+6t^{10}+8t^{12}+10t^{14}+t^{15}\\ &\quad +13t^{16}+t^{17}+16t^{18}+2t^{19}+20t^{20}+3t^{21}+24t^{22}\\ &\quad +4t^{23}+29t^{24}+6t^{25}+34t^{26}+8t^{27}+40t^{28}\\ &\quad +10t^{29}+47t^{30}+\cdots.\end{aligned}$$

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2. Weight Enumerators and Siegel Modular Forms. In this section we recall coding theory and Siegel modular forms. For the details we refer to [1], [15] for coding theory and to [3] for Siegel modular forms. Let $\mathbf{F}_2$ be the field of two elements. A linear code (a code for short) of length n is a subspace of $\mathbf{F}_2^n$. The vector space $\mathbf{F}_2^n$ equips with the inner product $x \cdot y = \sum x_i y_i$. We define the weight $wt(v)$ of a vector $v \in \mathbf{F}_2^n$ by the number of the non-zero coordinates of v. We shall define special classes of codes. If a code C coincides with its dual code $C^\perp = \{x \in \mathbf{F}_2^n | (x,y) = 0,\ {}^\forall y \in C\}$, it is called self-dual. We observe that the dimension of the self-dual code is a half of its length. If the weight of any element in C is divisible by 4, it is called doubly-even. In this manuscript we will focus on the self-dual doubly-even codes. We shall next define a homogeneous polynomial of the code which is on the title of this paper. The weight enumerator $W_C^{(g)}$ of the code C in genus g is defined by

$$W_C^{(g)} = W_C^{(g)}(x_a : a \in \mathbf{F}_2^g) = \sum_{v_1,\dots,v_g} \prod_{a \in \mathbf{F}_2^g} x_a^{n_a(v_1,\dots,v_g)},$$

where $n_a(v_1,\dots,v_g)$ denotes the number of i such that $a = (v_{1i},\dots,v_{gi})$. If we need the ordering of the elements of $\mathbf{F}_2^g$, we fix $\mathbf{F}_2^g = \{0\cdots 00,\ 0\cdots 01,\ 0\cdots 10,\dots,1\cdots 1\}$. We sometimes use the symbols $x,y,z,\dots$ instead of the x_a's for simplicity. The weight enumerator in genus 1 is interpreted as

$$W_C^{(1)} = \sum_{v \in C} x^{n-wt(v)} y^{wt(v)},$$

where n denotes the length of the code C. In this case the weight enumerator of a self-dual doubly-even code is a symmetric polynomial in the variables x,y. The examples are

$$\begin{aligned} W_{e_8}^{(1)} &= x^8 + 14x^4y^4 + y^8, \\ W_{g_{24}}^{(1)} &= x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24}, \end{aligned}$$

where e_8 denotes the extended Hamming code and g_{24} the extended Golay code. We omit the definitions of $e_8,\ g_{24}$ as well as d_n^+ appearing below and refer to the references cited above. In the case when $g=2$ the weight enumerator of a self-dual doubly-even code is also symmetric in the variables x,y,z,w. We have

$$\begin{aligned} W_{e_8}^{(2)} &= (8) + 14(4,4) + 168(2,2,2,2), \\ W_{g_{24}}^{(2)} &= (24) + 759(16,8) + 2576(12,12) + 212520(12,4,4,4) + 340032(10,6,6,2) \\ &\quad + 22770(8,8,8) + 1275120(8,8,4,4) + 4080384(6,6,6,6), \end{aligned}$$

where $(8) = x^8+y^8+z^8+w^8,\ (12,4,4,4) = x^{12}y^4z^4w^4 + x^4y^{12}z^4w^4 + x^4y^4z^{12}w^4 + x^4y^4z^4w^{12}$, etc. For an arbitrary positive integer $n,\ n \equiv 0 \pmod 8$, we have

$$W_{d_n^+}^{(2)} = \frac{1}{2^2} \sum_{\beta,\gamma \in \mathbf{F}_2^2} \left(\sum_{\alpha \in \mathbf{F}_2^2} (-1)^{\alpha\cdot\beta} x_{\alpha+\gamma} x_\alpha \right)^{n/2}.$$

We note that, for $g \geq 3$, the weight enumerator of a self-dual doubly-even code is not symmetric in general. We shall next view the weight enumerator from invariant theory of some finite group. Let H_g $(g \geq 1)$ be a finite subgroup of $GL(2^g, \mathbf{C})$ generated by

$$\left(\frac{1+i}{2}\right)^g \left((-1)^{a\cdot b}\right)_{a,b \in \mathbf{F}_2^g}, \qquad \mathrm{diag}\left(i^{S[a]}; a \in \mathbf{F}_2^g\right),\ S = {}^tS \in \mathrm{Mat}_{g\times g}(\mathbf{Z}),$$

where $A[B] = {}^tBAB$ for matrices A, B of suitable sizes. H_g has a normal subgroup $N_g \cong \mathbf{Z}/4\mathbf{Z} \star 2_+^{1+2g}$ such that $H_g/N_g \cong Sp(g, \mathbf{F}_2)$, where $\star$ denotes the central product and 2_+^{1+2g} the extra special 2-group of order $1+2g$ of "+" type. The finite group H_g has an order $2^{g^2+2g+2}(4^g - 1)(4^{g-1}-1)\cdots(4-1)$. We define another finite group G_g which is generated by H_g and the primitive eighth root of unity. The group G_g contains H_g as a subgroup of index 2. We have defined two finite groups so far. The group which directly concerns the weight enumerators is G_g. Indeed the weight enumerator of any self-dual doubly-even code is invariant under the action of G_g. Moreover the invariant ring of G_g can be generated by the weight enumerators of the self-dual doubly-even codes for any g (*cf.* [4], [6], [2], [5], [17], [11], [21]). Therefore we may regard the invariant ring $\mathbf{C}[x_a; a \in \mathbf{F}_2^g]^{G_g}$ as the ring of weight enumerators of the self-dual doubly-even codes in genus g.

Igusa's homomorphism is, under some condition, one from the ring $A(\Gamma_g)$ of Siegel modular forms to the ring $S(2, 2g+2)$ of projective invariants of a binary form of degree $2g+2$ (see [7]). We recall that that the ring $A(\Gamma_g)$ is the graded ring generated by holomorphic functions ψ on the Siegel upper-half space $\mathfrak{S}_g$ in genus g satisfying the functional equation

$$\psi(M \cdot \tau) = \det(c\tau + d)^k \cdot \psi(\tau)$$

for every M in $\Gamma_g = Sp(g, \mathbf{Z})$ (plus a condition at infinity for $g = 1$). In order to construct Siegel modular forms, we introduce the theta-constants. Theta-constants $\theta_{m'm''}(\tau)$ are defined by

$$\theta_{m'm''}(\tau) = \sum_{p \in \mathbf{Z}^g} \exp 2\pi\sqrt{-1}\left(\frac{1}{2}\tau\left[p + \frac{m'}{2}\right] + {}^t\left(p + \frac{m'}{2}\right)\frac{m''}{2}\right),$$

in which m' and m'' are the column vectors in $\mathbf{R}^g$. If we put $f_{m'}(\tau) = \theta_{m'0}(2\tau)$, the Broué-Enguehard map Th is defined by $x_a \mapsto f_a$. A modular form is called a cusp form if it is in the kernel of Siegel's Φ-operator which maps a modular form of genus g to a modular form of genus $g-1$.

The structures of the invariant rings and of Siegel modular forms in small genera are known. We shall describe the cases $g = 1,\ 2$. First suppose that $g = 1$. The groups $H_1,\ G_1$ are finite unitary reflection groups of order 96, 192, respectively (No.8, No.9 in the list of [19]). We have

$$\mathbf{C}[x, y]^{H_1} = \mathbf{C}[W_{e_8}^{(1)}, h_{12}^{(1)}], \qquad \mathbf{C}[x, y]^{G_1} = \mathbf{C}[W_{e_8}^{(1)},\ W_{g_{24}}^{(1)}],$$

where

$$h_{12}^{(1)} = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}.$$

The dimension formulae of these invariant rings are given by

$$\frac{1}{(1-t^8)(1-t^{12})}, \quad \frac{1}{(1-t^8)(1-t^{24})}.$$

The map Th induces the isomorphisms[2] $\mathbf{C}[x, y]^{H_1} \xrightarrow{\cong} A(\Gamma_1)$ and $\mathbf{C}[x, y]^{G_1} \xrightarrow{\cong} A(\Gamma_1)^{(4)}$. Here we remark that $A(\Gamma_1) = A(\Gamma_1)^{(2)}$. The invariant ring $\mathbf{C}[x, y]^{G_1}$ is a subring of $\mathbf{C}[x, y]^{H_1}$ and we observe that

$$W_{g_{24}}^{(1)} = 11 \cdot 2^{-1}3^{-2}(W_{e_8}^{(1)})^3 + 7 \cdot 2^{-1}3^{-2}(h_{12}^{(1)})^2.$$

[2] If S is a graded ring composed of homogeneous parts S_k with k running over non-negative integers and if d is a positive integer, then $S^{(d)} = \oplus_{k \geq 0} S_{dk}$.

The isomorphisms above are given by

$$\begin{aligned}
Th(W_{e_8}^{(1)}) &= \phi_4(\omega),\\
Th(h_{12}^{(1)}) &= \phi_6(\omega),\\
Th(W_{g_{24}}^{(1)}) &= 11\cdot 2^{-1}3^{-2}(\phi_4(\omega))^3 + 7\cdot 2^{-1}3^{-2}(\phi_6(\omega))^2,
\end{aligned}$$

where $\phi_k(\omega)$ denotes the normalized Eisenstein series of weight k: $\phi_k(\omega) = 1+\cdots$.

We shall next discuss the case when $g=2$. The group H_2 is a finite unitary reflection group of order 46080, No.31 in the list of [19]. The invariant ring of H_2 is generated by the four elements $W_{e_8}^{(2)}, W_{g_{24}}^{(2)}$,

$$\begin{aligned}
h_{12}^{(2)} &= (12) - 33(8,4) + 330(4,4,4) + 792(6,2,2,2),\\
F_{20} &= (20) - 19(16,4) - 336(14,2,2,2) - 494(12,8) + 716(12,4,4)\\
&\quad + 1038(8,8,4) + 7632(10,6,2,2) + 106848(6,6,6,2) + 129012(8,4,4,4).
\end{aligned}$$

The dimension formula of this ring is

$$\frac{1}{(1-t^8)(1-t^{12})(1-t^{20})(1-t^{24})} = 1+t^8+t^{12}+t^{16}+2t^{20}+3t^{24}+2t^{28}+4t^{32}+4t^{36}+5t^{40}+\cdots.$$

The group G_2 contains H_2 by index 2 and is not a finite unitary reflection group. The invariant ring is generated by $W_{e_8}^{(2)}, W_{g_{24}}^{(2)}, W_{d_{24}^+}^{(2)}, W_{d_{32}^+}^{(2)}$ and $W_{d_{40}^+}^{(2)}$. The four elements $W_{e_8}^{(2)}, W_{g_{24}}^{(2)}, W_{d_{24}^+}^{(2)}, W_{d_{40}^+}^{(2)}$ are algebraically independent and the square of $W_{d_{32}^+}^{(2)}$ is written by the polynomial in $W_{e_8}^{(2)}, W_{g_{24}}^{(2)}$, $W_{d_{24}^+}^{(2)}, W_{d_{40}^+}^{(2)}$ as follows:

$$\begin{aligned}
(W_{d_{32}^+}^{(2)})^2 &= -113\cdot 32621\cdot 3^{-4}5^{-1}7^{-2}41^{-1}(W_{e_8}^{(2)})^8\\
&\quad - 2^8 60289\cdot 3^{-4}5^{-1}7^{-2}11^{-1}41^{-1}(W_{e_8}^{(2)})^5 W_{g_{24}}^{(2)}\\
&\quad + 2^4 821477\cdot 3^{-4}5^{-1}7^{-1}11^{-1}41^{-1}(W_{e_8}^{(2)})^5 W_{d_{24}^+}^{(2)}\\
&\quad + 2\cdot 751\cdot 3^{-2}7^{-1}41^{-1}(W_{e_8}^{(2)})^4 W_{d_{32}^+}^{(2)}\\
&\quad - 2^9 11^2\cdot 3^{-3}5^{-1}7^{-1}41^{-1}(W_{e_8}^{(2)})^3 W_{d_{40}^+}^{(2)}\\
&\quad + 2^{14}163\cdot 3^{-4}7^{-2}11^{-2}41^{-1}(W_{e_8}^{(2)})^2 (W_{g_{24}}^{(2)})^2\\
&\quad + 2^{11}73\cdot 79\cdot 3^{-4}7^{-1}11^{-2}41^{-1}(W_{e_8}^{(2)})^2 W_{g_{24}}^{(2)} W_{d_{24}^+}^{(2)}\\
&\quad - 2^6 107\cdot 499\cdot 3^{-4}11^{-2}41^{-1}(W_{e_8}^{(2)})^2 (W_{d_{24}^+}^{(2)})^2\\
&\quad - 2^8 389\cdot 3^{-2}7^{-1}11^{-1}41^{-1}W_{e_8}^{(2)} W_{g_{24}}^{(2)} W_{d_{32}^+}^{(2)}\\
&\quad + 2^4 5\cdot 197\cdot 3^{-2}11^{-1}41^{-1}W_{e_8}^{(2)} W_{d_{24}^+}^{(2)} W_{d_{32}^+}^{(2)}\\
&\quad + 2^{12}3^{-1}5^{-1}7^{-1}41^{-1}W_{g_{24}}^{(2)} W_{d_{40}^+}^{(2)}\\
&\quad + 2^9 3^{-1}5^{-1}41^{-1}W_{d_{24}^+}^{(2)} W_{d_{40}^+}^{(2)}.
\end{aligned}$$

This was given in [21]. The dimension formula of this invariant ring $\mathbf{C}[x,y,z,w]^{G_2}$ is

$$\frac{1+t^{32}}{(1-t^8)(1-t^{24})^2(1-t^{40})} = 1+t^8+t^{16}+3t^{24}+4t^{32}+5t^{40}+8t^{48}+10t^{56}+12t^{64}+\cdots.$$

The elements $W^{(2)}_{d^+_{24}}, W^{(2)}_{d^+_{32}}, W^{(2)}_{d^+_{40}}$ can be written by the generators of $\mathbf{C}[x,y,z,w]^{H_2}$ as follows:

$$
\begin{aligned}
W^{(2)}_{d^+_{24}} &= 11^2 3^{-2} 7^{-1} (W^{(2)}_{e_8})^3 + 2\cdot 3^{-2} (h^{(2)}_{12})^2 - 2^3 7^{-1} W^{(2)}_{g_{24}},\\
W^{(2)}_{d^+_{32}} &= 43\cdot 53\cdot 3^{-4} 7^{-1} (W^{(2)}_{e_8})^4 + 2^4 5\cdot 23\cdot 3^{-5} 11^{-1} W^{(2)}_{e_8} (h^{(2)}_{12})^2\\
&\quad - 2^6 43\cdot 3^{-2} 7^{-1} 11^{-1} W^{(2)}_{e_8} W^{(2)}_{g_{24}} + 2^6 3^{-5} h^{(2)}_{12} F_{20},\\
W^{(2)}_{d^+_{40}} &= 3\cdot 19\cdot 7^{-1} (W^{(2)}_{e_8})^5 + 2\cdot 5\cdot 7\cdot 557\cdot 3^{-7} 11^{-1} (W^{(2)}_{e_8})^2 (h^{(2)}_{12})^2\\
&\quad - 2^3 5\cdot 19\cdot 7^{-1} 11^{-1} (W^{(2)}_{e_8})^2 W^{(2)}_{g_{24}} + 2^6 5^2 3^{-7} W^{(2)}_{e_8} h^{(2)}_{12} F_{20} + 2^2 5\cdot 41\cdot 3^{-7} F_{20}^2.
\end{aligned}
$$

We give a comment on the paper [9]. In that paper, Maschke determined the invariant ring of some finite group G. G is a subgroup of $SL(4,\mathbf{C})$ and has an order 46080 which is the same as our H_2. G is a subgroup of our G_2, which is of an order $2\cdot 46080 = 92160$. H_2 is generated by three elements

$$
\left(\frac{1+i}{2}\right)^2 \left((-1)^{a\cdot b}\right)_{a,b\in\mathbf{F}_2^2},\ \mathrm{diag}\left(1,1,\sqrt{-1},\sqrt{-1}\right),\ \mathrm{diag}\left(1,1,1,-1\right),
$$

and G_2 by H_2 and $\frac{1+i}{\sqrt{2}}$, while G is generated by

$$
\left(\frac{1+i}{2}\right)^2 \left((-1)^{a\cdot b}\right)_{a,b\in\mathbf{F}_2^2},\ \frac{1+i}{\sqrt{2}}\cdot\mathrm{diag}\left(1,1,\sqrt{-1},\sqrt{-1}\right),\ \frac{1+i}{\sqrt{2}}\cdot\mathrm{diag}\left(1,1,1,-1\right).
$$

The dimension formula of $\mathbf{C}[x,y,z,w]^G$ is given by

$$
\frac{1+t^{32}+t^{60}+t^{92}}{(1-t^8)(1-t^{24})^2(1-t^{40})}.
$$

From the dimension formulae, for example, we can read off the differences among the invariant rings of the said groups.

We continue our discussion on our case. We shall recall that $A(\Gamma_2)$ is generated over $\mathbf{C}$ by five elements and they are[3]

$$
\begin{aligned}
2^2\cdot\psi_4 &= \sum (\theta_m)^8,\\
2^2\cdot\psi_6 &= \sum_{\text{syzygous}} \pm (\theta_{m_1}\theta_{m_2}\theta_{m_3})^4,\\
-2^{14}\cdot\chi_{10} &= \prod (\theta_m)^2,\\
2^{17}3\cdot\chi_{12} &= \sum (\theta_{m_1}\theta_{m_2}\cdots\theta_{m_6})^4,\\
2^{39}5^3\sqrt{-1}\cdot\chi_{35} &= \left(\prod\theta_m\right)\left(\sum_{\text{azygous}} \pm (\theta_{m_1}\theta_{m_2}\theta_{m_3})^{20}\right).
\end{aligned}
$$

In the second symmetrization, the monomial $(\theta_{m_1}\theta_{m_2}\theta_{m_3})^4$ with ${}^t m_1 = (0,0,0,0)$, ${}^t m_2 = (0,0,0,1)$, ${}^t m_3 = (0,0,1,0)$ has $+1$ as its coefficient. In the definition of χ_{12}, the summation is extended over fifteen complements of syzygous quadruples. In the definition of χ_{35}, the symmetrization of $\pm(\theta_{m_1}\theta_{m_2}\theta_{m_3})^{20}$ is taken by the stabilizer of $\prod\theta_m$ in $Sp(2,\mathbf{Z})$ modulo

[3] χ_{35} is not used in Section 3.

the stabilizer of $(\theta_{m_1}\theta_{m_2}\theta_{m_3})^{20}$ with ${}^t m_1 = (0,0,0,0), {}^t m_2 = (0,0,0,1), {}^t m_3 = (0,1,0,0)$. The Broué-Enguehard map gives rise the following:

$$\begin{aligned}
Th(W_{e_8}^{(2)}) &= \psi_4,\\
Th(h_{12}^{(2)}) &= \psi_6,\\
Th(F_{20}) &= \psi_4\psi_6 + 2^{12}3^4\chi_{10},\\
Th(W_{g_{24}}^{(2)}) &= 11\cdot 2^{-1}3^{-2}\psi_4^3 + 7\cdot 2^{-1}3^{-2}\psi_6^2 - 2^{10}3^2 7\cdot 11\chi_{12}.
\end{aligned}$$

These can be obtained by comparing the Fourier coefficients (*cf.* [16], [14], [13]). There have been extensive studies on Fourier coefficients of Siegel modular forms, however, in our case we do not need a deep theory of Fourier coefficients. Since there is a misprint in the definition of F_4 in [8] (corrected in [10]), we reproduce the formulae which are useful for our computations of Fourier coefficients. In the case when $g = 1$, we shall use ω instead of τ. If we put

$$F_0(r) = \sum_{p=1}^{\infty} r^{p^2}, \qquad F_1(r) = \sum_{p=1}^{\infty} r^{(p-1/2)^2},$$

in which $r = \exp \pi\sqrt{-1}\omega$, then we have

$$\theta_{00}(\omega) = 1 + 2F_0(r), \quad \theta_{01}(\omega) = 1 + 2F_0(-r), \quad \theta_{10}(\omega) = 2F_1(r).$$

In the case when $g = 2$ if we put

$$\begin{aligned}
F_0(r_1,r_2) &= F_0(r_1) + F_0(r_2) + \sum_{p_1,p_2=1}^{\infty} A_{p_1,p_2} r_1^{p_1^2} r_2^{p_2^2},\\
F_1(r_1,r_2) &= F_1(r_2) + \sum_{p_1,p_2=1}^{\infty} B_{p_1,p_2} r_1^{p_1^2} r_2^{(p_2-1/2)^2},\\
F_2(r_1,r_2) &= F_1(r_2,r_1),\\
F_3(r_1,r_2) &= \sum_{p_1,p_2=1}^{\infty} C_{p_1,p_2} r_1^{(p_1-1/2)^2} r_2^{(p_2-1/2)^2},\\
F_4(r_1,r_2) &= \sum_{p_1,p_2=1}^{\infty} D_{p_1,p_2} r_1^{(p_1-1/2)^2} r_2^{(p_2-1/2)^2},
\end{aligned}$$

in which $r_1 = \exp \pi\sqrt{-1}\tau_1, r_2 = \exp \pi\sqrt{-1}\tau_2, q_{12} = \exp 2\pi\sqrt{-1}\tau_{12}$, and

$$\begin{aligned}
A_{p_1,p_2} &= q_{12}^{p_1p_2} + q_{12}^{-p_1p_2},\\
B_{p_1,p_2} &= q_{12}^{p_1(p_2-1/2)} + q_{12}^{-p_1(p_2-1/2)},\\
C_{p_1,p_2} &= q_{12}^{(p_1-1/2)(p_2-1/2)} + q_{12}^{-(p_1-1/2)(p_2-1/2)},\\
D_{p_1,p_2} &= (-1)^{p_1+p_2-1} q_{12}^{(p_1-1/2)(p_2-1/2)} + (-1)^{p_1-p_2} q_{12}^{-(p_1-1/2)(p_2-1/2)},
\end{aligned}$$

then we will have

$$\begin{aligned}
&\theta_{0000}(\tau) = 1 + 2F_0(r_1, r_2), && \theta_{0001}(\tau) = 1 + 2F_0(r_1, -r_2),\\
&\theta_{0010}(\tau) = 1 + 2F_0(-r_1, r_2), && \theta_{0011}(\tau) = 1 + 2F_0(-r_1, -r_2),\\
&\theta_{0100}(\tau) = 2F_1(r_1, r_2), && \theta_{0110}(\tau) = 2F_1(-r_1, r_2),\\
&\theta_{1000}(\tau) = 2F_2(r_1, r_2), && \theta_{1001}(\tau) = 2F_2(r_1, -r_2),\\
&\theta_{1100}(\tau) = 2F_3(r_1, r_2), && \theta_{1111}(\tau) = 2F_4(r_1, r_2).
\end{aligned}$$

Cusp forms can be written by Eisenstein series as follows:

$$\begin{aligned}
\chi_{10} &= -43867 \cdot 2^{-12}3^{-5}5^{-2}7^{-1}53^{-1} \left(\psi_4\psi_6 - \psi_{10}\right),\\
\chi_{12} &= 131 \cdot 593 \cdot 2^{-13}3^{-7}5^{-3}7^{-2}337^{-1} \left(3^2 7^2 \psi_4^3 + 2 \cdot 5^3 \psi_6^2 - 691\psi_{12}\right).
\end{aligned}$$

We note that there is a misprint in the formula of χ_{10} at p.102 in [16].

3. The Broué-Enguehard map and Igusa's homomorphism. Before proceeding to Igusa's homomorphism studied in [7], we go back to the invariant rings $S(2,4)$, $S(2,6)$. In addition to the generators of them given in Section 1, we give the different generators in irrational forms. If we decompose a binary form into linear factors as

$$\sum_{i=0}^{n} \binom{n}{i} u_i x^{n-i} y^i = u_0 \prod_{i=1}^{n} (x - \xi_i y),$$

then we have

$$-\binom{n}{1}\frac{u_1}{u_0} = \sum_{i=1}^{n} \xi_i,\ \binom{n}{2}\frac{u_2}{u_0} = \sum_{i<j} \xi_i\xi_j,\ \cdots,\ (-1)^n\binom{n}{n}\frac{u_n}{u_0} = \prod_{i=1}^{n} \xi_i.$$

Preparing this, we consider each case separately[4]. We put $P_{2g+2}(x) = u_0(x-\xi_1)(x-\xi_2)\cdots(x-\xi_{2g+2})$. Suppose that $g = 1$. In [7], Igusa takes $I_2(P_4(x)), I_3(P_4(x))$ as the generators of $S(2,4)$, given as follows:

$$\begin{aligned}
I_2(P_4(x)) &= u_0^2 \sum_{\text{three}} (12)^2(34)^2\\
&= u_0^2 \left\{(12)^2(34)^2 + (13)^2(24)^2 + (14)^2(23)^2\right\},\\
I_3(P_4(x)) &= u_0^3 \sum_{\text{six}} (12)^2(34)^2(13)(24)\\
&= u_0^3 \left\{(12)^2(34)^2\left\{(13)(24) + (14)(23)\right\} + (13)^2(24)^2\{(12)(34) - (14)(23)\} + \right.\\
&\qquad \left.(14)^2(23)^2\{-(12)(34) - (13)(24)\}\right\}.
\end{aligned}$$

Here (ij) is an abridged notation for $\xi_i - \xi_j$. We already gave the generators of $S(2,4)$ in Section 1 and these two sets of the generators are related each other by

$$I_2(P_4(x)) = 2^3 3P, \qquad I_3(P_4(x)) = 2^4 3^3 Q.$$

[4]We note that the case when $g = 1$ is roughly sketched in [12].

Igusa's homomorphism ρ gives the isomorphism $\rho : A(\Gamma_1) \xrightarrow{\cong} S(2,4)$, where

$$\rho(\psi_4(\tau)) = 2^{-1} I_2(P_4(x)),$$
$$\rho(\psi_6(\tau)) = 2^{-1} I_3(P_4(x)).$$

Combining two maps Th and ρ to denote $\widetilde{\rho}$, we have the isomorphism $\mathbf{C}[x,y]^{H_1} \xrightarrow{\cong} S(2,4)$ given by

$$\widetilde{\rho}(W_{e_8}^{(1)}) = 2^{-1} I_2(P_4(x)),$$
$$\widetilde{\rho}(h_{12}^{(1)}) = 2^{-1} I_3(P_4(x)).$$

If we use $P,\ Q$ defined in Section 1, then

$$\widetilde{\rho}(W_{e_8}^{(1)}) = 2^2 3P,$$
$$\widetilde{\rho}(h_{12}^{(1)}) = 2^3 3^3 Q.$$

We also get

$$\mathbf{C}[x,y]^{G_1} \xrightarrow{\cong} S(2,4)^{(2)},$$

where

$$\begin{aligned}\widetilde{\rho}(W_{g_{24}}^{(1)}) &= 11 \cdot 2^{-4} 3^{-2} I_2(P_4(x))^3 + 7 \cdot 2^{-3} 3^{-2} I_3(P_4(x))^2 \\ &= 2^5 3 \left(11P^3 + 3^3 7Q^2\right).\end{aligned}$$

We shall next consider the case when $g = 2$. In [7] the following elements[5] are used as the generators of $S(2,6)$.

$$\begin{aligned}
A(P_6(x)) &= u_0^2 \sum_{\text{fifteen}} (12)^2(34)^2(56)^2, \\
B(P_6(x)) &= u_0^4 \sum_{\text{ten}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2, \\
C(P_6(x)) &= u_0^6 \sum_{\text{sixty}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2, \\
D(P_6(x)) &= u_0^{10} \underbrace{(12)^2(13)^2 \cdots (56)^2}_{\text{fifteen}}, \\
E(P_6(x)) &= u_0^{15} \prod_{\text{fifteen}} \det \begin{pmatrix} 1 & \xi_1 + \xi_2 & \xi_1\xi_2 \\ 1 & \xi_3 + \xi_4 & \xi_3\xi_4 \\ 1 & \xi_5 + \xi_6 & \xi_5\xi_6 \end{pmatrix}.
\end{aligned}$$

There hold the following relations.

$$\begin{aligned}
A(P_6(x)) &= -2^4 3 \cdot 5 J_2, \\
B(P_6(x)) &= 2^2 3^4 5 \left(J_2^2 - 2^2 5^2 J_4\right), \\
C(P_6(x)) &= 2^3 3^2 5 \left(-2^4 13 J_2^3 + 2^6 3^2 5^2 J_2 J_4 + 5^3 J_6\right), \\
D(P_6(x)) &= 2^3 3^3 \left(2^2 571 J_2^5 + 2^5 3^2 5^3 J_2^3 J_4 + 2^2 5^4 J_2^2 J_6 - 2^6 3^4 5^5 J_2 J_4^2 + 2^3 3^2 5^5 J_4 J_6 - 3^3 5^5 J_{10}\right), \\
E(P_6(x)) &= 2^2 3^9 5^{10} J_{15}.
\end{aligned}$$

[5] We note that we do not need E in this paper.

We do not need the formula of E^2 in this paper, however, since it is not contained in [7], we give the explicit formula of E^2. This is derived from the formula of J_{15}^2 in the appendix, or directly.

$$\begin{aligned}
E^2 = (1/2^{11}3^9)\big(&A^7B^4 - 2^23A^6B^3C - 2^23^5A^6B^2D \\
&+ 2\cdot 3\cdot 13A^5B^5 + 2\cdot 3^3A^5B^2C^2 + 2^33^6A^5BCD + 2^23^{10}A^5D^2 \\
&- 2^23^237A^4B^4C - 2^23^4239A^4B^3D - 2^23^3A^4BC^3 - 2^23^7A^4C^2D \\
&- 3\cdot 53A^3B^6 + 2\cdot 3^45\cdot 11A^3B^3C^2 + 2^93^57A^3B^2CD + 2^53^75^211A^3BD^2 \\
&+ 3^4A^3C^4 + 2^63^3A^2B^5C + 2^43^4457A^2B^4D - 2^63^317A^2B^2C^3 \\
&- 2^43^65\cdot 53A^2BC^2D - 2^73^85^3A^2CD^2 + 2^45AB^7 - 2^53^37AB^4C^2 \\
&- 2^63^55\cdot 61AB^3CD - 2^63^75^329AB^2D^2 + 2^43^437ABC^4 + 2^63^75^2AC^3D \\
&- 2^73B^6C - 2^93^4B^5D + 2^83^3B^3C^3 + 2^93^65^2B^2C^2D \\
&+2^93^85^4BCD^2 - 2^73^5C^5 + 2^{11}3^95^5D^3\big).
\end{aligned}$$

Igusa used this to get the expression for χ_{35}^2 in [7]. Igusa's ρ-homomorphism is given by

$$\begin{aligned}
\rho(\psi_4) &= 2^{-2}B, \\
\rho(\psi_6) &= 2^{-3}(AB - 3C) \\
&= 3^35\left(-2^319J_2^3 + 2^53^35^2J_2J_4 - 5^3J_6\right), \\
\rho(\chi_{10}) &= -2^{-14}D, \\
\rho(\chi_{12}) &= 2^{-17}3^{-1}AD \\
&= 3^35\cdot 2^{-10}(-2^2571J_2^6 - 2^53^25^3J_2^4J_4 - 2^25^4J_2^3J_6 + 2^63^45^5J_2^2J_4^2 \\
&\quad - 2^33^25^5J_2J_4J_6 + 3^35^5J_2J_{10}), \\
\rho(\chi_{35}) &= -2^{-39}\sqrt{-1}D^2E.
\end{aligned}$$

This homomorphism is injective (Theorem 5 in [7]). If we shall denote by $\widetilde{\rho}$ the composition of the Broué-Enguehard map and Igusa's ρ-homomorphism, we will have

$$\begin{aligned}
\widetilde{\rho}(W_{e_8}^{(2)}) &= \rho(\psi_4) \\
&= 2^{-2}B, \\
\widetilde{\rho}(h_{12}^{(2)}) &= \rho(\psi_6) \\
&= 2^{-3}(AB - 3C), \\
\widetilde{\rho}(F_{20}) &= \rho(\psi_4\psi_6 + 2^{12}3^4\chi_{10}) \\
&= 2^{-5}\left(AB^2 - 3BC - 2^33^4D\right) \\
&= 3^7\left(-2^4523J_2^5 + 2^75^353J_2^3J_4 - 5^413J_2^2J_6 - 2^83^35^5J_2J_4^2 - 2^25^511J_4J_6 + 2\cdot 3^35^5J_{10}\right), \\
\widetilde{\rho}(W_{g_{24}}^{(2)}) &= \rho(11\cdot 2^{-1}3^{-2}\psi_4^3 + 7\cdot 2^{-1}3^{-2}\psi_6^2 - 2^{10}3^27\cdot 11\chi_{12}) \\
&= 2^{-7}3^{-2}\left(7A^2B^2 - 2\cdot 3\cdot 7ABC - 3^37\cdot 11AD + 11B^3 + 3^27C^2\right) \\
&= 3^45\cdot 2^{-1}(2064323J_2^6 - 2^23^45^317\cdot 397J_2^4J_4 + 2^35^47\cdot 71J_2^3J_6 \\
&\quad + 2^43^55^51223J_2^2J_4^2 - 2^43^55^57J_2J_4J_6 - 2\cdot 3^45^57\cdot 11J_2J_{10} \\
&\quad - 2^63^65^811J_4^3 + 5^77J_6^2).
\end{aligned}$$

Using these formulae, we know the $\widetilde{\rho}$ image of the generators (except $W_{e_8}^{(2)},\ W_{g_{24}}^{(2)}$) of $\mathbf{C}[x,y,z,w]^{G_2}$ as follows:

$$\begin{aligned}
\widetilde{\rho}(W_{d_{24}^+}^{(2)}) &= \widetilde{\rho}(11^2 3^{-2} 7^{-1} (W_{e_8}^{(2)})^3 + 2\cdot 3^{-2} (h_{12}^{(2)})^2 - 2^3 7^{-1} W_{g_{24}}^{(2)}) \\
&= 2^{-6}3^{-2}(-2A^2B^2 + 2^2 3ABC + 2^2 3^3 11AD + 11B^3 - 2\cdot 3^2 C^2) \\
&= 3^4 5(-409\cdot 1549 J_2^6 - 2^2 3^3 5^4 59 J_2^4 J_4 - 2^7 5^4 13 J_2^3 J_6 + 2^4 3^5 5^5 463 J_2^2 J_4^2 \\
&\quad - 2^6 3^3 5^5 J_2 J_4 J_6 + 2^3 3^4 5^5 11 J_2 J_{10} - 2^6 3^6 5^8 11 J_4^3 - 2\cdot 5^7 J_6^2), \\
\widetilde{\rho}(W_{d_{32}^+}^{(2)}) &= \widetilde{\rho}(43\cdot 53\cdot 3^{-4} 7^{-1} (W_{e_8}^{(2)})^4 + 2^4 5\cdot 23\cdot 3^{-5} 11^{-1} W_{e_8}^{(2)} (h_{12}^{(2)})^2 \\
&\quad - 2^6 43\cdot 3^{-2} 7^{-1} 11^{-1} W_{e_8}^{(2)} W_{g_{24}}^{(2)} + 2^6 3^{-5} h_{12}^{(2)} F_{20}) \\
&= 2^{-8}3^{-3}(-2^4 A^2 B^3 + 2^5 3AB^2 C + 2^5 3^5 ABD + 43B^4 - 2^4 3^2 BC^2 + 2^9 3^3 CD) \\
&= 3^5 5(-286322081 J_2^8 + 2^4 3^2 5^2 31\cdot 59\cdot 5477 J_2^6 J_4 - 2^{11} 5^3 17\cdot 83 J_2^5 J_6 \\
&\quad + 2^5 3^6 5^6 7\cdot 43 J_2^4 J_4^2 + 2^9 3^3 5^5 13\cdot 61 J_2^3 J_4 J_6 + 2^6 3^3 5^5 7\cdot 233 J_2^3 J_{10} \\
&\quad - 2^8 3^6 5^7 37\cdot 239 J_2^2 J_4^3 - 2^4 5^7 13 J_2^2 J_6^2 - 2^{11} 3^5 5^7 J_2 J_4^2 J_6 - 2^8 3^5 5^7 167 J_2 J_4 J_{10} \\
&\quad + 2^8 3^8 5^{11} 43 J_4^4 + 2^6 3^2 5^8 41 J_4 J_6^2 - 2^7 3^3 5^8 J_6 J_{10}), \\
\widetilde{\rho}(W_{d_{40}^+}^{(2)}) &= \widetilde{\rho}(3\cdot 19\cdot 7^{-1} (W_{e_8}^{(2)})^5 + 2\cdot 5\cdot 7\cdot 557\cdot 3^{-7} 11^{-1} (W_{e_8}^{(2)})^2 (h_{12}^{(2)})^2 \\
&\quad - 2^3 5\cdot 19\cdot 7^{-1} 11^{-1} (W_{e_8}^{(2)})^2 W_{g_{24}}^{(2)} + 2^6 5^2 3^{-7} W_{e_8}^{(2)} h_{12}^{(2)} F_{20} + 2^2 5\cdot 41\cdot 3^{-7} F_{20}^2) \\
&= 2^{-10}3^{-2}(-2\cdot 5A^2B^4 + 2^2 3\cdot 5AB^3C + 2^2 3^4 5AB^2 D + 19B^5 \\
&\quad - 2\cdot 3^2 5B^2C^2 + 2^6 3^4 5BCD + 2^8 3^3 5\cdot 41 D^2) \\
&= 3^7 5(-4129\cdot 5298991 J_2^{10} + 2^2 3^2 5^3 157\cdot 8119907 J_2^8 J_4 - 2^6 5^4 17\cdot 25171 J_2^7 J_6 \\
&\quad - 2^5 3^5 5^5 13\cdot 409\cdot 3121 J_2^6 J_4^2 + 2^6 3^2 5^5 73\cdot 44887 J_2^5 J_4 J_6 + 2^3 3^3 5^5 397\cdot 1867 J_2^5 J_{10} \\
&\quad - 2^7 3^6 5^8 67\cdot 631 J_2^4 J_4^3 + 2\cdot 5^8 11\cdot 1103 J_2^4 J_6^2 - 2^9 3^4 5^8 23609 J_2^3 J_4^2 J_6 \\
&\quad - 2^6 3^5 5^8 22571 J_2^3 J_4 J_{10} + 2^8 3^8 5^{10} 55901 J_2^2 J_4^4 + 2^4 3^2 5^9 5639 J_2^2 J_4 J_6^2 \\
&\quad - 2^4 3^3 5^9 733 J_2^2 J_6 J_{10} + 2^{10} 3^6 5^{10} 379 J_2 J_4^3 J_6 + 2^7 3^7 5^{10} 17\cdot 163 J_2 J_4^2 J_{10} \\
&\quad - 2^{10} 3^{11} 5^{14} 19 J_4^5 - 2^5 3^4 5^{10} 23\cdot 181 J_4^2 J_6^2 + 2^6 3^5 5^{10} 61 J_4 J_6 J_{10} + 2^4 3^6 5^{10} 41 J_{10}^2).
\end{aligned}$$

We observe that the ρ images of $A(\Gamma_2)^{(2)},\ A(\Gamma_2)^{(4)}$ are strictly smaller than $S(2,6)^{(2)},\ S(2,6)^{(4)}$, respectively. On the other hand, it is known that the Broué-Enguehard map induces the isomorphisms $\mathbf{C}[x,y,z,w]^{H_2} \cong A(\Gamma_2)^{(2)}$ and $\mathbf{C}[x,y,z,w]^{G_2} \cong A(\Gamma_2)^{(4)}$. Therefore the $\widetilde{\rho}$ images of the rings $\mathbf{C}[x,y,z,w]^{H_2},\ \mathbf{C}[x,y,z,w]^{G_2}$ are strictly smaller than $S(2,6)^{(2)},\ S(2,6)^{(4)}$, respectively. Summing up,

Theorem. *Let $\widetilde{\rho}$ be the composition of the Broué-Enguehard map and Igusa's ρ-homomorphism.*

(1) *In the case when $g=1$, $\widetilde{\rho}$ gives rise to the isomorphisms from $\mathbf{C}[x,y]^{H_1}$ onto $S(2,4)$, and from $\mathbf{C}[x,y]^{G_1}$ onto $S(2,4)^{(2)}$.*

(2) *In the case when $g=2$, $\widetilde{\rho}$ transforms injectively $\mathbf{C}[x,y,z,w]^{H_2}$ and $\mathbf{C}[x,y,z,w]^{G_2}$ into $S(2,6)$. The $\widetilde{\rho}$ images of these invariant rings are strictly smaller than $S(2,6)^{(2)},\ S(2,6)^{(4)}$, respectively.*

The explicit $\widetilde{\rho}$-images of the generators of each invariant ring are given above in two ways each.

We give remarks.

(1) Since Igusa's ρ homomorphism increases the weight or the degree by a $\frac{1}{2}g$ ratio, our $\widetilde{\rho}$ increases the degree by a $\frac{1}{4}g$ ratio. This remark holds in the arbitrary genus g. Here we note that Siegel modular forms we are considering are always of even weights.

(2) The weight enumerator of a code has non-negative integers as its coefficients(in the arbitrary genus). We shall consider the case when $g = 1$. The $\widetilde{\rho}$ image of the weight enumerator has negative coefficients as the polynomials in $\mathbf{C}[u_0, u_1, u_2, u_3, u_4]$ in general. For example, we have

$$\begin{aligned}
\widetilde{\rho}(W^{(1)}_{e_8}) &= 2^2 3(u_0u_4 - 4u_1u_3 + 3u_2^2),\\
\widetilde{\rho}(W^{(1)}_{g_{24}}) &= 2^5 3 \left(11u_0^3u_4^3 - 132u_0^2u_1u_3u_4^2 + 288u_0^2u_2^2u_4^2 - 378u_0^2u_2u_3^2u_4 + 189u_0^2u_3^4\right.\\
&\quad - 378u_0u_1^2u_2u_4^2 + 906u_0u_1^2u_3^2u_4 - 36u_0u_1u_2^2u_3u_4 - 756u_0u_1u_2u_3^3\\
&\quad - 81u_0u_2^4u_4 + 378u_0u_2^3u_3^2 + 189u_1^4u_4^2 - 756u_1^3u_2u_3u_4 - 704u_1^3u_3^3\\
&\quad \left. + 378u_1^2u_2^3u_4 + 2340u_1^2u_2^2u_3^2 - 1944u_1u_2^4u_3 + 486u_2^6\right).
\end{aligned}$$

It would be interesting if we interpret this from coding theoretical or combinatorial point of view.

(3) We mention the paper [20] in which Shioda discussed the close relationship of the ring $S(3, 4)$ of projective invariants to the invariant theory for the Weyl groups $W(E_7)$ and $W(E_6)$. We omit the details.

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Appendix. We give the generators of $S(2,6)$ from [18]. We also give the expression for J_{15}^2.

$$
\begin{aligned}
J_2 &= u_0u_6 - 6u_1u_5 + 15u_2u_4 - 10u_3^2,\\
J_4 &= \det\begin{pmatrix} u_0 & u_1 & u_2 & u_3\\ u_1 & u_2 & u_3 & u_4\\ u_2 & u_3 & u_4 & u_5\\ u_3 & u_4 & u_5 & u_6 \end{pmatrix},\\
J_6 &= \det\begin{pmatrix} b_0 & b_1 & b_2\\ b_1 & b_2 & b_3\\ b_2 & b_3 & b_4 \end{pmatrix},\\
J_{10} &= u_0\, c^3 - 6u_1\, bc^2 + 3u_2(ac+4b^2)c - 4u_3(3abc+2b^3) + 3u_4\, a(ac+4b^2) - 6u_5\, a^2b + u_6\, a^3,\\
J_{15} &= \det\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_4\\ c_1 & c_2 & c_3 & c_4 & c_5\\ c_2 & c_3 & c_4 & c_5 & c_6\\ c_3 & c_4 & c_5 & c_6 & c_7\\ c_4 & c_5 & c_6 & c_7 & c_8 \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
b_0 &= 6(u_0u_4 - 4u_1u_3 + 3u_2^2),\\
b_1 &= 3(u_0u_5 - 3u_1u_4 + 2u_2u_3),\\
b_2 &= u_0u_6 - 9u_2u_4 + 8u_3^2,\\
b_3 &= 3(u_1u_6 - 3u_2u_5 + 2u_3u_4),\\
b_4 &= 6(u_2u_6 - 4u_3u_5 + 3u_4^2),\\
a &= 2(u_0u_2u_6 - 3u_0u_3u_5 + 2u_0u_4^2 - u_1^2u_6 + 3u_1u_2u_5 - u_1u_3u_4 - 3u_2^2u_4 + 2u_2u_3^2),\\
b &= u_0u_3u_6 - u_0u_4u_5 - u_1u_2u_6 - 8u_1u_3u_5 + 9u_1u_4^2 + 9u_2^2u_5 - 17u_2u_3u_4 + 8u_3^3,\\
c &= 2(u_0u_4u_6 - u_0u_5^2 - 3u_1u_3u_6 + 3u_1u_4u_5 + 2u_2^2u_6 - u_2u_3u_5 - 3u_2u_4^2 + 2u_3^2u_4),\\
c_0 &= 8(u_0^2u_5 - 5u_0u_1u_4 + 2u_0u_2u_3 + 8u_1^2u_3 - 6u_1u_2^2),\\
c_1 &= u_0^2u_6 + 2u_0u_1u_5 - 19u_0u_2u_4 + 8u_0u_3^2 - 6u_1^2u_4 + 44u_1u_2u_3 - 30u_2^3,\\
c_2 &= 2(u_0u_1u_6 - 2u_0u_2u_5 - 2u_0u_3u_4 - 3u_1u_2u_4 + 16u_1u_3^2 - 10u_2^2u_3),\\
c_3 &= u_0u_2u_6 - 4u_0u_3u_5 - 2u_0u_4^2 + 2u_1^2u_6 - 6u_1u_2u_5 + 24u_1u_3u_4 - 15u_2^2u_4,\\
c_4 &= 4(-u_0u_4u_5 + u_1u_2u_6 + 3u_1u_4^2 - 3u_2^2u_5),\\
c_5 &= -u_0u_4u_6 - 2u_0u_5^2 + 4u_1u_3u_6 + 6u_1u_4u_5 + 2u_2^2u_6 - 24u_2u_3u_5 + 15u_2u_4^2,\\
c_6 &= 2(-u_0u_5u_6 + 2u_1u_4u_6 + 2u_2u_3u_6 + 3u_2u_4u_5 - 16u_3^2u_5 + 10u_3u_4^2),\\
c_7 &= -u_0u_6^2 - 2u_1u_5u_6 + 19u_2u_4u_6 + 6u_2u_5^2 - 8u_3^2u_6 - 44u_3u_4u_5 + 30u_4^3,\\
c_8 &= 8(-u_1u_6^2 + 5u_2u_5u_6 - 2u_3u_4u_6 - 8u_3u_5^2 + 6u_4^2u_5).
\end{aligned}
$$

$$
\begin{aligned}
J_{15}^2 = & -2^7 3^{-10} J_2^{15} + 2^9 7 \cdot 3^{-8} J_2^{13} J_4 - 2^7 37 \cdot 3^{-12} J_2^{12} J_6 \\
& - 2^{11} 7 \cdot 3^{-5} J_2^{11} J_4^2 + 2^{15} 3^{-9} J_2^{10} J_4 J_6 + 2^7 3^{-7} J_2^{10} J_{10} \\
& + 2^{13} 5 \cdot 7 \cdot 3^{-4} J_2^9 J_4^3 - 2^8 29 \cdot 3^{-12} J_2^9 J_6^2 - 2^{11} 5 \cdot 3^{-4} J_2^8 J_4^2 J_6 \\
& - 2^9 5 \cdot 3^{-5} J_2^8 J_4 J_{10} - 2^{15} 5 \cdot 7 \cdot 3^{-2} J_2^7 J_4^4 + 2^{10} 11 \cdot 3^{-8} J_2^7 J_4 J_6^2 \\
& + 2^7 3^{-6} J_2^7 J_6 J_{10} + 2^{16} 5 \cdot 11 \cdot 3^{-6} J_2^6 J_4^3 J_6 + 2^{12} 5 \cdot 3^{-3} J_2^6 J_4^2 J_{10} \\
& - 2^8 7 \cdot 3^{-11} J_2^6 J_6^3 + 2^{17} 3 \cdot 7 J_2^5 J_4^5 - 2^{11} 3^{-3} J_2^5 J_4^2 J_6^2 \\
& - 2^{10} 5 \cdot 3^{-5} J_2^5 J_4 J_6 J_{10} + 2^5 3^{-3} J_2^5 J_{10}^2 - 2^{15} 5 \cdot 17 \cdot 3^{-3} J_2^4 J_4^4 J_6 \\
& - 2^{14} 5 \cdot 3^{-1} J_2^4 J_4^3 J_{10} + 2^{12} 3^{-8} J_2^4 J_4 J_6^3 + 2^7 3^{-6} J_2^4 J_6^2 J_{10} \\
& - 2^{19} 3^2 7 J_2^3 J_4^6 + 2^{15} 31 \cdot 3^{-6} J_2^3 J_4^3 J_6^2 + 2^{11} 11 \cdot 3^{-3} J_2^3 J_4^2 J_6 J_{10} \\
& - 2^8 3^{-1} J_2^3 J_4 J_{10}^2 - 2^7 13 \cdot 3^{-12} J_2^3 J_6^4 + 2^{20} J_2^2 J_4^5 J_6 \\
& + 2^{15} 3 \cdot 5 J_2^2 J_4^4 J_{10} - 2^{11} 3^{-5} J_2^2 J_4^2 J_6^3 - 2^9 5 \cdot 3^{-5} J_2^2 J_4 J_6^2 J_{10} \\
& + 2^5 3^{-3} J_2^2 J_6 J_{10}^2 + 2^{21} 3^4 J_2 J_4^7 - 2^{15} 7 \cdot 3^{-3} J_2 J_4^4 J_6^2 \\
& - 2^{15} 3^{-1} J_2 J_4^3 J_6 J_{10} + 2^9 3 J_2 J_4^2 J_{10}^2 + 2^9 3^{-9} J_2 J_4 J_6^4 \\
& + 2^7 3^{-7} J_2 J_6^3 J_{10} - 2^{19} 7 J_4^6 J_6 - 2^{17} 3^3 J_4^5 J_{10} \\
& - 2^{13} 3^{-6} J_4^3 J_6^3 + 2^{11} 3^{-3} J_4^2 J_6^2 J_{10} + 2^7 3^{-1} J_4 J_6 J_{10}^2 \\
& - 2^7 3^{-12} J_6^5 - 2^5 J_{10}^3.
\end{aligned}
$$