Basic Invariants of the Complex Reflection Group No.34 Constructed by Conway and Sloane

Manabu Oura* Jiro Sekiguchi†

Abstract

This paper studies the basic invariants, constructed by Conway and Sloane, of the complex reflection group numbered as 34 in the list of Shephard-Todd [14].

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1 Introduction

In this paper, we treat the complex reflection group of number 34 in the list of Shephard and Todd [14] (denote this group by ST34 in this paper) which has order $6^5 \cdot 7!$. The degrees of its basic invariants are 6,12,18,24,30,42. If $x = (x_1, x_2, \cdots, x_6)$ is a linear coordinate of the 6 dimensional representation space of ST34, the basic invariants are polynomials of x. Conway and Sloane [5] constructed such basic invariants which are denoted by μ_j $(j = 6, 12, 18, 24, 30, 42)$. It is hard to write them down as polynomials of x because of their lengthy. The main purpose of this paper is to write them in a reasonable size. We explain our idea to accomplish the purpose briefly. Let $G(3, 3, 6)$ be the imprimitive complex reflection group of rank 6. Since $G(3, 3, 6)$ is a subgroup of ST34, each of μ_j $(j = 6, 12, 18, 24, 30, 42)$ is written as a polynomial of the basic invariants of $G(3, 3, 6)$, say $p_3, p_6, p_9, p_{12}, p_{15}, s_6$. It is easy to write $p_3, p_6, p_9, p_{12}, p_{15}, s_6$ down as polynomials of x. Along this idea, we succeeded to write μ_j $(j = 6, 12, 18, 24, 30, 42)$ as polynomials of $p_3, p_6, p_9, p_{12}, p_{15}, s_6$ in a reasonable size. Though it follows from the definition that each μ_j is invariant by ST34, we shall give an alternative proof of the invariance by use of symmetric polynomials of six letters (see Theorem 1).

The discriminant of ST34 is expressed as a polynomial of μ_j $(j = 6, 12, 18, 24, 30, 42)$. It is known (cf. [1]) that the discriminant of a complex reflection group G is expressed as the determinant of the Saito matrix of G. In ST34 case, Terao and Enta [15] proposed an algorithm to compute the Saito matrix for the basic invariants μ_j $(j = 6, 12, 18, 24, 30, 42)$ and along this line, Bessis and Michel actually computed it explicitly (unpublished). On the other hand, Kato, Mano and Sekiguchi ([7], [8]) formulated the notion of the flat

*Institute of Science and Engineering, Kanazawa University, Japan.
Email address: oura@se.kanazawa-u.ac.jp
†Emeritus Professor, Tokyo University of Agriculture and Technology, Japan.
Email address: jinomino@yahoo.co.jp

structure which is a generalization of Frobenius manifold structure. Applying it to a complex reflection group G, one can construct the Saito matrix for the flat coordinate of G when G satisfies some conditions on invariants. In ST34 case, the Saito matrix corresponding to a potential vector field of 6 variables whose weights are the same as the degrees of the basic invariants of ST34 is constructed in [7] and the determinant of this Saito matrix is expected to be the discriminant of ST34 if we regard the variables as appropriate basic invariants. Theorem 2 says that this is actually valid. Theorem 2 itself is the same as the result in [12] but its proof is different from that given in [12] (see Remark 2 in the main text).

In the rest of this paper, we shall treat two topics related with the group ST34. One is the correspondence between the totality of minimal vectors of the Coxeter-Todd lattice [5] and that of pseudo-reflections of ST34. We shall describe a natural 6-1 correspondence between them. The other is concerned with the restriction of the basic invariants μ_j ($j = 6, 12, 18, 24, 30, 42$) to the representation space of the group ST33 which is the complex reflection group No.33 in [14]. The basic invariants of ST33 were first constructed by Burkhardt (cf. [9]).

This paper is organized as follows. In section 2, we mention the Coxeter-Todd lattice, its minimal vectors and the basic invariants μ_j ($j = 6, 12, 18, 24, 30, 42$) of ST34 constructed by Conway and Sloane. In section 3, we first introduce polynomials m_j ($j = 1, 2, 3, 4, 5, 7$) written as polynomials of basic invariants $p_3, p_6, p_9, p_{12}, p_{15}, s_6$ of the complex reflection group $G(3, 3, 6)$. Then in Theorem 1 we shall show that m_j coincides with μ_{6j} up to a constant. In section 4, we mention two results on invariants and a discriminant of ST34 due to Terao, Enta, Bessis and Michel. In section 5, we describe the discriminant of ST34 in terms of the flat coordinates of ST34 (cf. [7], [8], [12]) In section 6, we shall treat two topics on ST34. One is concerned with the correspondence between the totality of minimal vectors of the Coxeter-Todd lattice and that of hyperplanes fixed by the pseudo-reflections of ST34. The other is the relationship between the basic invariants of the group ST33 and those of ST34 constructed by Conway and Sloane. In the Appendix, we give the explicit relationship between Burkhardt invariants and our invariants.

We finally mention that the softwares Mathematica, Maple and Magma [2] are used to obtain the results in this paper.

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2 The basic invariants by Conway-Sloane

By the classification of complex reflection groups of Shephard-Todd [14], it is known that there are three infinite series of complex reflection groups and thirty-four sporadic ones numbered as $1, 2, \ldots, 37$. In this paper we focus our attention on the group No.34 which we denote by ST34. The basic invariants of ST34 are constructed in Theorem 10 of [5] and they are denoted by μ_j ($j = 6, 12, 18, 24, 30, 42$). We now explain one of their constructions briefly. Let $\Lambda^{(3)}$ be the Coxeter-Todd lattice introduced at p.424 of [5]. Then its automorphism group is nothing else but ST34. Throughout this paper, we write $\omega = e^{2\pi i/3}$ and $\theta = \omega - \overline{\omega} = \sqrt{-3}$ without any comment. The minimal vectors of $\Lambda^{(3)}$ consist of the vectors

$$\pm(\omega^a\theta, -\omega^b\theta, 0, 0, 0, 0),\ a, b \in \{0, 1, 2\},$$
$$(\omega^a, \omega^b, \omega^c, \omega^d, \omega^e, \omega^f),\ a, b, c, d, e, f \in \{0, 1, 2\},\ a + b + c + d + e + f \equiv 0 \pmod 3$$

where all possible coordinate changes are considered in the first type. There are $2 \cdot 3^2 \cdot \binom{6}{2} = 270$ minimal vectors of the first type, whereas $2 \cdot 3^5 = 486$ minimal vectors of the second type. In total, we have 756 minimal vectors of $\Lambda^{(3)}$. We set

$$\mu_k = \sum (v_1x_1 + \cdots + v_6x_6)^k,\ k = 0, 1, 2\ldots,$$

where $v = (v_1, \ldots, v_6)$ runs through 756 minimal vectors of $\Lambda^{(3)}$. Theorem 10 in [5] says that the invariant ring of ST34 is generated by

$$\mu_6, \mu_{12}, \mu_{18}, \mu_{24}, \mu_{30}, \mu_{42}.$$

We collect the basic properties of ST34. There are 126 pseudo-reflections in ST34. The hyperplanes fixed by the pseudo-reflections of ST34 are

$$\begin{array}{rr}
45 & x_i - \omega^a x_j = 0,\ a \in \{0, 1, 2\}, \\
30 & x_1 + x_2 + x_3 + x_4 + \omega x_5 + \omega^2 x_6 = 0, \\
20 & x_1 + x_2 + x_3 + \omega(x_4 + x_5 + x_6) = 0, \\
30 & x_1 + x_2 + \omega(x_3 + x_4) + \omega^2(x_5 + x_6) = 0, \\
1 & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0
\end{array}$$

where the number on the left side denotes the cardinality of each type. ST34 can be generated by transformations $P_1, P_2, P_3, Q_1, R_1, R_2$ given at p.298 in [14]. The hyperplanes fixed by the generators are

$$\begin{aligned}
&P_1 : x_2 - x_3 = 0, \\
&P_2 : x_3 - x_4 = 0, \\
&P_3 : x_4 - x_5 = 0, \\
&Q_1 : x_1 - x_2 = 0, \\
&R_1 : x_1 - \omega x_2 = 0, \\
&R_2 : x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0.
\end{aligned}$$

The 7-th power of the matrix $R_2R_1Q_1P_1P_2P_3$ given at p.299 of [14] is $-\omega I_6$. The center Z_{34} of ST34 is of order 6 (cf. Table VII of [14]) and $-\omega I_6$ centralizes ST34. As a consequence, $-\omega I_6$ generates Z_{34}.

3 From $G(3, 3, 6)$ to ST34

Since the group $G(3, 3, 6)$ is a subgroup[1] of ST34 and since these two groups are of the same rank, every invariant by ST34 is a polynomial of the basic invariants of $G(3, 3, 6)$.

[1]The group $G(3, 3, 6)$ is generated by the transformations on $(x_1, \ldots, x_6)$: (1) the permutations on $x_1, \ldots, x_6$, (2) $x_i \mapsto \epsilon_i x_i$ $(i = 1, \ldots, 6)$, $\epsilon_i^3 = 1$, $\epsilon_1 \ldots \epsilon_6 = 1$. The reflection corresponding to $x_5 - x_6 = 0$ is given as the product of $(R_2R_1Q_1P_1P_2)^9$ and $-I_6$. The resulting matrix is an element of ST34.

It is known (p.284 of [14]) that as the basic invariants of $G(3,3,6)$, we may take p_{3j} $(j = 1,2,3,4,5)$ and s_6 defined by

$$p_{3j} = x_1^{3j} + x_2^{3j} + x_3^{3j} + x_4^{3j} + x_5^{3j} + x_6^{3j} \; (j = 1,2,3,4,5), \; s_6 = x_1x_2x_3x_4x_5x_6.$$

Let m_j $(j = 1,2,3,4,5,7)$ be the polynomials of p_j $(j = 3,6,9,12,15)$ and s_6 defined by the following identities. These m_j's are indeed invariants of ST34 as we will show below.

$$\begin{aligned}
m_1 &= -5p_3^2 + 6p_6 - 180s_6, \\
m_2 &= \tfrac{1}{17}(-10125p_{12} + 1925p_3^4 - 10395p_3^2p_6 + 6237p_6^2 + 12375p_3p_9 + 51975p_3^2s_6 - 40095p_6s_6 + 935550s_6^2), \\
m_3 &= \tfrac{(1}{3644}(-1088916048p_{15}p_3 + 1401597270p_{12}p_3^2 - 52286707p_3^6 - 3171150p_{12}p_6 + 498415005p_3^4p_6 \\
&\quad -723353895p_3^2p_6^2 + 9550629p_6^3 - 961539480p_3^3p_9 + 936512280p_3p_6p_9 - 16804260p_9^2 + 2817424620p_{12}s_6 \\
&\quad -804053250p_3^4s_6 + 3721617900p_3^2p_6s_6 - 1435481190p_6^2s_6 - 4300536240p_3p_9s_6 - 13894040160p_3^2s_6^2 \\
&\quad +8931882960p_6s_6^2 - 99578658960s_6^3), \\
m_4 &= \tfrac{1}{984160}(616868762940p_{12}^2 + 13316119811280p_{15}p_3^3 - 16940634571080p_{12}p_3^4 + 594592089355p_3^8 \\
&\quad -21567358775952p_{15}p_3p_6 + 28072219631940p_{12}p_3^2p_6 - 6811339217958p_3^6p_6 - 368171695980p_{12}p_6^2 \\
&\quad +18219805403580p_3^4p_6^2 - 14058186924870p_3^2p_6^3 + 98666056665p_6^4 + 9579454289856p_{15}p_9 \\
&\quad -13482219282840p_{12}p_3p_9 + 11937182241204p_3^5p_9 - 34871337186840p_3^3p_6p_9 + 24742879985340p_3p_6^2p_9 \\
&\quad +9126295239600p_3^2p_9^2 - 8204834872080p_6p_9^2 + 417113860322016p_{15}p_3s_6 - 559068289484760p_{12}p_3^2s_6 \\
&\quad +24970204832364p_3^6s_6 + 17333883551160p_{12}p_6s_6 - 219949713575460p_3^4p_6s_6 + 302052795820380p_3^2p_6^2s_6 \\
&\quad -8679490869060p_6^3s_6 + 401008538842560p_3^3p_9s_6 - 376118508300480p_3p_6p_9s_6 + 1336769866080p_9^2s_6 \\
&\quad -732306533056560p_{12}s_6^2 + 311680371562560p_3^4s_6^2 - 1202195718884160p_3^2p_6s_6^2 + 386420052498480p_6^2s_6^2 \\
&\quad +1238625892183680p_3p_9s_6^2 + 3635549644541760p_3^2s_6^3 - 1960702790302080p_6s_6^3 + 14488157891797920s_6^4),
\end{aligned}$$

$$\begin{aligned}
m_5 &= \tfrac{1}{106288160}(-8558297914454928000p_{12}{}^2p_3{}^2 + 2089201018632300p_{12}{}^2p_6 + 1165675074483701520p_{12}p_{15}p_3 \\
&\quad +138192145506496980p_{12}p_3{}^6 - 877484405515505400p_{12}p_3{}^4p_6 + 1379987318783043000p_{12}p_3{}^3p_9 \\
&\quad +903656701052469000p_{12}p_3{}^2p_6{}^2 - 1000122875204029200p_{12}p_3p_6p_9 - 906698664553500p_{12}p_6{}^3 \\
&\quad +580661494716600p_{12}p_9{}^2 - 386759386157508288p_{15}{}^2 - 103164357358799784p_{15}p_3{}^5 \\
&\quad +631029397643881200p_{15}p_3{}^3p_6 - 965383545286463760p_{15}p_3{}^2p_9 - 618935794330053240p_{15}p_3p_6{}^2 \\
&\quad +662798218317617520p_{15}p_6p_9 - 3721032235241428p_3{}^{10} + 56929735554738525p_3{}^8p_6 \\
&\quad -100670803599215340p_3{}^7p_9 - 269791048596035610p_3{}^6p_6{}^2 + 754513827309400680p_3{}^5p_6p_9 \\
&\quad +455456084098977900p_3{}^4p_6{}^3 - 540975355278457500p_3{}^4p_9{}^2 - 1258582004466149700p_3{}^3p_6{}^2p_9 \\
&\quad -237876161937857550p_3{}^2p_6{}^4 + 824140299141199800p_3{}^2p_6p_9{}^2 + 530150266143331200p_3p_6{}^3p_9 \\
&\quad -157057655983200p_3p_9{}^3 + 168811773953535p_6{}^5 - 285007424484525300p_6{}^2p_9{}^2) \\
&\quad -\tfrac{2349}{5314408}(3336381283590p_{12}{}^2 - 90991170059715p_{12}p_3{}^4 + 132473392247970p_{12}p_3{}^2p_6 \\
&\quad -60473056848750p_{12}p_3p_9 - 3642116633685p_{12}p_6{}^2 + 69747109280496p_{15}p_3{}^3 \\
&\quad -96145773383952p_{15}p_3p_6 + 39062903267592p_{15}p_9 + 3483778065289p_3{}^8 - 38013477459228p_3{}^6p_6 \\
&\quad +65401030889973p_3{}^5p_9 + 94878525690540p_3{}^4p_6{}^2 - 173998725667650p_3{}^3p_6p_9 \\
&\quad -66633212619360p_3{}^2p_6{}^3 + 42238214518500p_3{}^2p_9{}^2 + 110872962731835p_3p_6{}^2p_9 + 987000196815p_6{}^4 \\
&\quad -32583765485700p_6p_9{}^2)s_6 \\
&\quad -\tfrac{221240565}{5314408}(-14731974630p_{12}p_3{}^2 + 693432090p_{12}p_6 + 10397417904p_{15}p_3 + 842268971p_3{}^6 \\
&\quad -6659025525p_3{}^4p_6 + 11239209720p_3{}^3p_9 + 8453844855p_3{}^2p_6{}^2 - 9972788040p_3p_6p_9 - 348684345p_6{}^3 \\
&\quad +86329800p_9{}^2)s_6^2 \\
&\quad +\tfrac{98415}{1328602}(7805322022518p_{12} - 4961331496841p_3{}^4 + 16060185954756p_3{}^2p_6 - 14673183959034p_3p_9 \\
&\quad -4333560899931p_6{}^2)s_6^3 \\
&\quad -\tfrac{231176835}{664301}(9206527958p_3{}^2 - 4219165557p_6)s_6^4 \\
&\quad -\tfrac{5437195842606312330}{664301}s_6^5,
\end{aligned}$$

$$
\begin{aligned}
m_7 \;=\; & \tfrac{1}{619872782080}(-681557026068425577785745{p_3}^{14}+1432379563699839285857067p_6{p_3}^{12} \\
& -2557436825642562030969900p_9{p_3}^{11}-10698541080341814802394019{p_6}^2{p_3}^{10} \\
& +3582500028944048652957174p_{12}{p_3}^{10}-2730902888436717092575680p_{15}{p_3}^9 \\
& +32708776402929577905944640p_6p_9{p_3}^9+34924283387125205511721185{p_6}^3{p_3}^8 \\
& -25654418121315705916250700{p_9}^2{p_3}^8-41680220280677434506011850p_{12}p_6{p_3}^8 \\
& +31369465880845892613346176p_{15}p_6{p_3}^7-123740484999420235384965240{p_6}^2p_9{p_3}^7 \\
& +69769318329654728499822240p_{12}p_9{p_3}^7-52762146031720621233390555{p_6}^4{p_3}^6 \\
& -47257748766395678591117100{p_{12}}^2{p_3}^6+126780633297174166548934500p_{12}{p_6}^2{p_3}^6 \\
& +130162353083938619632641600p_6{p_9}^2{p_3}^6-52307455008394026247756320p_{15}p_9{p_3}^6 \\
& -34758789936079286842349760{p_9}^3{p_3}^5-92627028581352964521141312p_{15}{p_6}^2{p_3}^5 \\
& +70714813144026703680064512p_{12}p_{15}{p_3}^5+186354004242240630334859040{p_6}^3p_9{p_3}^5 \\
& -268191568191852515860009920p_{12}p_6p_9{p_3}^5+33299590235273255328597225{p_6}^5{p_3}^4 \\
& -127770635569767758534767500p_{12}{p_6}^3{p_3}^4-26421815125729745638761600{p_{15}}^2{p_3}^4 \\
& -201898474896088367649617400{p_6}^2{p_9}^2{p_3}^4+113298473516080679802360000p_{12}{p_9}^2{p_3}^4 \\
& +122348893127056155271952100{p_{12}}^2p_6{p_3}^4+190543330015178442011760960p_{15}p_6p_9{p_3}^4 \\
& +87812086740565495801127040p_{15}{p_6}^3{p_3}^3+62665614050169799982476800p_6{p_9}^3{p_3}^3 \\
& -74023416302855483255097600p_{15}{p_9}^2{p_3}^3-167964296041394365510972800p_{12}p_{15}p_6{p_3}^3 \\
& -106338866778530075623833900{p_6}^4p_9{p_3}^3-111554371285879313232176400{p_{12}}^2p_9{p_3}^3 \\
& +262063973509069744833861600p_{12}{p_6}^2p_9{p_3}^3-5126918175165039183578025{p_6}^6{p_3}^2 \\
& +27931623849212462954949150p_{12}{p_6}^4{p_3}^2-23152418725131358790400{p_9}^4{p_3}^2 \\
& +29692604870039550095987400{p_{12}}^3{p_3}^2-50201877703761207864827100{p_{12}}^2{p_6}^2{p_3}^2 \\
& +103065371624630753152920000{p_6}^3{p_9}^2{p_3}^2-115014969795605791696092000p_{12}p_6{p_9}^2{p_3}^2 \\
& +56973925496711539138136832{p_{15}}^2p_6{p_3}^2-169930072089259625905241760p_{15}{p_6}^2p_9{p_3}^2 \\
& +135574470385751288032364160p_{12}p_{15}p_9{p_3}^2-14845481613081683657896320p_{15}{p_6}^4p_3 \\
& -27945506509393387416667200{p_6}^2{p_9}^3p_3+34586384532087307046400p_{12}{p_9}^3p_3 \\
& +51824289113657057141222400p_{12}p_{15}{p_6}^2p_3+65877259573354459565260800p_{15}p_6{p_9}^2p_3 \\
& -44231372298721494546912000{p_{12}}^2p_{15}p_3+12594866818006221239484000{p_6}^5p_9p_3 \\
& -44075507458043013535144800p_{12}{p_6}^3p_9p_3-38836159574544740438311680{p_{15}}^2p_9p_3 \\
& +37742148440266625813304000{p_{12}}^2p_6p_9p_3+841205579632152292755{p_6}^7-6381502411588805286450p_{12}{p_6}^5 \\
& +20837200939432768800000p_6{p_9}^4+12591883060088162966100{p_{12}}^2{p_6}^3-18536934715459277725440p_{15}{p_9}^3 \\
& +16391367254585859259256064p_{12}{p_{15}}^2-10659165988259339409769344{p_{15}}^2{p_6}^2 \\
& -7683466994926210007484300{p_6}^4{p_9}^2-8464031877608794242000{p_{12}}^2{p_9}^2 \\
& +11976082225174674297633600p_{12}{p_6}^2{p_9}^2+4971887755839338531400{p_{12}}^3p_6 \\
& +18094988745615635242320960p_{15}{p_6}^3p_9-28029510009872734558063680p_{12}p_{15}p_6p_9) \\
& +\tfrac{2583}{309936391040}(-826650327935043055861{p_3}^{12}+14425925885422918422282p_6{p_3}^{10} \\
& -25452001272005719299540p_9{p_3}^9-83351888440402255303005{p_6}^2{p_3}^8+35197343052145422459210p_{12}{p_3}^8 \\
& -26417038086987608202768p_{15}{p_3}^7+242911215262797035732640p_6p_9{p_3}^7+186506835419875414158900{p_6}^3{p_3}^6 \\
& -183741456439968965505360{p_9}^2{p_3}^6-298429989407564825623320p_{12}p_6{p_3}^6 \\
& +220420573635965479389936p_{15}p_6{p_3}^5-563000045054149040896440{p_6}^2p_9{p_3}^5 \\
& +493130102035689915584400p_{12}p_9{p_3}^5-161110222055187816091275{p_6}^4{p_3}^4 \\
& -328749745772938424739300{p_{12}}^2{p_3}^4+485097352985170111227300p_{12}{p_6}^2{p_3}^4 \\
& +429094779958543624758000p_6{p_9}^2{p_3}^4-361895253749323218243360p_{15}p_9{p_3}^4 \\
& -205828786208959390080000{p_9}^3{p_3}^3-347376551409240131304720p_{15}{p_6}^2{p_3}^3 \\
& +479349562682466181945440p_{12}p_{15}{p_3}^3+428618573732090041704000{p_6}^3p_9{p_3}^3 \\
& -673892039580382647561600p_{12}p_6p_9{p_3}^3+36297779413268324630250{p_6}^5{p_3}^2 \\
& -143166572523976223269800p_{12}{p_6}^3{p_3}^2-173568570666357873223296{p_{15}}^2{p_3}^2 \\
& -293344069145812931192400{p_6}^2{p_9}^2{p_3}^2+43066790405325599148000p_{12}{p_9}^2{p_3}^2 \\
& +141143057326507215360600{p_{12}}^2p_6{p_3}^2+469676819583053026119360p_{15}p_6p_9{p_3}^2 \\
& +90703842698655568839600p_{15}{p_6}^3p_3+19163765066700626496000p_6{p_9}^3p_3 \\
& -22962898216765380119040p_{15}{p_9}^2p_3-178484620941100228128480p_{12}p_{15}p_6p_3
\end{aligned}
$$

$$
\begin{aligned}
&-8102926971070013137700{p_6}^4p_9p_3 - 1916143554430030203720 0{p_{12}}^2p_9p_3\\
&+16920269086916363100840 0p_{12}{p_6}^2p_9p_3 - 4445415058421567587 5{p_6}^6 + 2367554395045022302 50p_{12}{p_6}^4\\
&-955347510651264 00{p_9}^4 + 2365498512383748570 00{p_{12}}^3 - 4139692103496753567 00{p_{12}}^2{p_6}^2\\
&+4386040199445219504120 0{p_6}^3{p_9}^2 - 1186977993500726650800 0p_{12}p_6{p_9}^2 + 535899541123073297560 32{p_{15}}^2p_6\\
&-9727858648475385318576 0p_{15}{p_6}^2p_9 + 1421941087109823484320 0p_{12}p_{15}p_9)s_6\\
&-\tfrac{637362999}{15496819552}(4857452670185866{p_3}^{10} - 6528237829576110 9p_6{p_3}^8 + 108907344194688000p_9{p_3}^7\\
&+260295982554842820{p_6}^2{p_3}^6 - 144443551320429948p_{12}{p_3}^6 + 105268487601706560p_{15}{p_3}^5\\
&-644181269743573344p_6p_9{p_3}^5 - 367603906096725030{p_6}^3{p_3}^4 + 384498967408122240{p_9}^2{p_3}^4\\
&+678555020839017240p_{12}p_6{p_3}^4 - 463955787849692736p_{15}p_6{p_3}^3 + 895430833203929760{p_6}^2p_9{p_3}^3\\
&-880017911185364640p_{12}p_9{p_3}^3 + 153022160508667650{p_6}^4{p_3}^2 + 472031454180039120{p_{12}}^2{p_3}^2\\
&-541859921711888940p_{12}{p_6}^2{p_3}^2 - 505515465619816320p_6{p_9}^2{p_3}^2 + 568858413963720960p_{15}p_9{p_3}^2\\
&+1078387051311360{p_9}^3p_3 + 333737005975634880p_{15}{p_6}^2p_3 - 573606450773307072p_{12}p_{15}p_3\\
&-298809060268406400{p_6}^3p_9p_3 + 519137518908140640p_{12}p_6p_9p_3 - 740481396190785{p_6}^5\\
&+2897609287598280p_{12}{p_6}^3 + 163131844841144064{p_{15}}^2 + 133116924334917600{p_6}^2{p_9}^2\\
&-769650476421600p_{12}{p_9}^2 - 2833333167373020{p_{12}}^2p_6 - 295206239616621696p_{15}p_6p_9)s_6^2\\
&-\tfrac{137781}{7748409776}(212849412401280527467{p_3}^8 - 205673769233333007991 2p_6{p_3}^6 + 330167579423351499783 0p_9{p_3}^5\\
&+448724675792067291621 0{p_6}^2{p_3}^4 - 431070373616947568424 0p_{12}{p_3}^4 + 310307544144292962556 8p_{15}{p_3}^3\\
&-754245841126396856994 0p_6p_9{p_3}^3 - 262455094398814898832 0{p_6}^3{p_3}^2 + 164863522620183734316 0{p_9}^2{p_3}^2\\
&+511230170861761511628 0p_{12}p_6{p_3}^2 - 325928932606666468972 8p_{15}p_6p_3 + 387057763297931032881 0{p_6}^2p_9p_3\\
&-214279423390894774590 0p_{12}p_9p_3 + 66893663401936554375{p_6}^4 + 193991306712427220580{p_{12}}^2\\
&-230654127137064281640p_{12}{p_6}^2 - 903663040115630763720p_6{p_9}^2 + 107361000213721818696 0p_{15}p_9)s_6^3\\
&-\tfrac{457570701}{3874204888}(393541072937010113{p_3}^6 - 2483580961309967805p_6{p_3}^4 + 3573657602659051360p_9{p_3}^3\\
&+2671680990475403505{p_6}^2{p_3}^2 - 4136783742478404450p_{12}{p_3}^2 + 2632941690092063712p_{15}p_3\\
&-2833712226159225840p_6p_9p_3 - 136991407212027225{p_6}^3 + 52735937047941600{p_9}^2 + 266860623034204470p_{12}p_6)s_6^4\\
&+\tfrac{125452159093170}{484275611}(-1416246125627{p_3}^4 + 3374374908123p_6{p_3}^2 - 2450724434568p_9p_3\\
&-673355559033{p_6}^2 + 1069099330203p_{12})s_6^5\\
&-\tfrac{9039811410}{484275611}(105868047331971014{p_3}^2 - 36630825385046211p_6)s_6^6\\
&-\tfrac{12664094659813992533357906 10}{484275611}s_6^7.
\end{aligned}
$$

The following theorem is the main result of this paper.

Theorem 1 *The polynomials m_j $(j = 1, 2, 3, 4, 5, 7)$ are invariants of* ST34.

Proof. The group ST34 can be generated by the pseudo-reflections $P_1, P_2, P_3, Q_1, R_1, R_2$. It is clear that $p_{3j}(j = 1, 2, 3, 4, 5)$ and s_6 are invariant under the action of P_1, P_2, P_3, Q_1, R_1 and so are m_j $(j = 1, 2, 3, 4, 5, 7)$.

As a consequence, it is sufficient to show that each of m_j is invariant by the action of R_2. For this purpose, we introduce the polynomials

$$q_j = \sum_{k=1}^{6} x_k^j \quad (j = 1, 2, 3, 4, 5).$$

Then p_{3j} $(j=1,2,3,4,5)$ are polynomials of q_j $(j=1,2,3,4,5)$ and s_6. Indeed, we have

$$
\begin{aligned}
p_3 &= q_3, \\
p_6 &= \tfrac{1}{120}(q_1^6 - 15q_1^4q_2 + 45q_1^2q_2^2 - 15q_2^3 + 40q_1^3q_3 - 120q_1q_2q_3 + 40q_3^2 - 90q_1^2q_4 + 90q_2q_4 + 144q_1q_5 - 720s_6), \\
p_9 &= \tfrac{1}{720}(q_1^9 - 9q_1^7q_2 - 27q_1^5q_2^2 + 135q_1^3q_2^3 + 33q_1^6q_3 + 45q_1^4q_2q_3 - 135q_1^2q_2^2q_3 - 135q_2^3q_3 + 120q_1^3q_3^2 - 360q_1q_2q_3^2 \\
&\quad +80q_3^3 - 54q_1^5q_4 - 270q_1^3q_2q_4 - 270q_1^2q_3q_4 + 270q_2q_3q_4 + 54q_1^4q_5 + 324q_1^2q_2q_5 + 162q_2^2q_5 + 432q_1q_3q_5 \\
&\quad +324q_4q_5 - 1080q_1^3s_6 - 3240q_1q_2s_6 - 2160q_3s_6), \\
p_{12} &= \tfrac{1}{43200}(q_1^{12} - 12q_1^{10}q_2 + 90q_1^8q_2^2 - 900q_1^6q_2^3 + 2025q_1^4q_2^4 + 80q_1^9q_3 - 720q_1^7q_2q_3 + 3600q_1^5q_2^2q_3 - 3600q_1^3q_2^3q_3 \\
&\quad +1680q_1^6q_3^2 - 3600q_1^4q_2q_3^2 + 3600q_1^2q_2^2q_3^2 - 3600q_2^3q_3^2 + 3200q_1^3q_3^3 - 9600q_1q_2q_3^3 + 1600q_3^4 - 45q_1^8q_4 + 180q_1^6q_2q_4 \\
&\quad -6750q_1^4q_2^2q_4 + 8100q_1^2q_2^3q_4 - 2025q_2^4q_4 - 2880q_1^5q_3q_4 + 7200q_1^3q_2q_3q_4 - 21600q_1q_2^2q_3q_4 - 7200q_1^2q_3^2q_4 \\
&\quad +7200q_2q_3^2q_4 - 16200q_1^2q_2q_4^2 + 8100q_2^2q_4^2 + 2700q_4^3 + 288q_1^7q_5 - 3456q_1^5q_2q_5 + 12960q_1^3q_2^2q_5 + 11520q_1^4q_3q_5 \\
&\quad -17280q_1^2q_2q_3q_5 + 11520q_1q_3^2q_5 - 17280q_1^3q_4q_5 + 25920q_1q_2q_4q_5 + 17280q_3q_4q_5 + 20736q_1^2q_5^2 + 10368q_2q_5^2 \\
&\quad -1440q_1^6s_6 - 64800q_1^2q_2^2s_6 - 57600q_1^3q_3s_6 - 57600q_3^2s_6 - 129600q_2q_4s_6 - 207360q_1q_5s_6 + 259200s_6^2), \\
p_{15} &= \tfrac{1}{1036800}(13q_1^{15} - 330q_1^{13}q_2 + 2565q_1^{11}q_2^2 - 4500q_1^9q_2^3 - 15525q_1^7q_2^4 + 44550q_1^5q_2^5 - 10125q_1^3q_2^6 + 895q_1^{12}q_3 \\
&\quad -12090q_1^{10}q_2q_3 + 22275q_1^8q_2^2q_3 + 78300q_1^6q_2^3q_3 - 165375q_1^4q_2^4q_3 - 20250q_1^2q_2^5q_3 + 10125q_2^6q_3 + 16000q_1^9q_3^2 \\
&\quad -36000q_1^7q_2q_3^2 - 64800q_1^5q_2^2q_3^2 + 108000q_1^3q_2^3q_3^2 + 108000q_1q_2^4q_3^2 + 24800q_1^6q_3^3 - 84000q_1^4q_2q_3^3 + 108000q_1^2q_2^2q_3^3 \\
&\quad -36000q_2^3q_3^3 + 32000q_1^3q_3^4 - 96000q_1q_2q_3^4 + 12800q_3^5 - 1710q_1^{11}q_4 + 19800q_1^9q_2q_4 - 2700q_1^7q_2^2q_4 \\
&\quad -178200q_1^5q_2^3q_4 + 101250q_1^3q_2^4q_4 - 58050q_1^8q_3q_4 - 16200q_1^6q_2q_3q_4 + 337500q_1^4q_2^2q_3q_4 + 81000q_1^2q_2^3q_3q_4 \\
&\quad -101250q_2^4q_3q_4 - 43200q_1^5q_3^2q_4 + 216000q_1^3q_2q_3^2q_4 - 432000q_1q_2^2q_3^2q_4 - 72000q_1^2q_3^3q_4 + 72000q_2q_3^3q_4 \\
&\quad +56700q_1^7q_4^2 + 113400q_1^5q_2q_4^2 - 121500q_1^3q_2^2q_4^2 + 40500q_1^4q_3q_4^2 - 405000q_1^2q_2q_3q_4^2 + 121500q_2^2q_3q_4^2 - 81000q_1^3q_4^3 \\
&\quad +81000q_3q_4^3 + 2196q_1^{10}q_5 - 22140q_1^8q_2q_5 - 33480q_1^6q_2^2q_5 + 243000q_1^4q_2^3q_5 + 72900q_1^2q_2^4q_5 - 24300q_2^5q_5 \\
&\quad +77760q_1^7q_3q_5 + 60480q_1^5q_2q_3q_5 - 302400q_1^3q_2^2q_3q_5 - 388800q_1q_2^3q_3q_5 + 216000q_1^4q_3^2q_5 - 432000q_1^2q_2q_3^2q_5 \\
&\quad -43200q_2^2q_3^2q_5 + 115200q_1q_3^3q_5 - 136080q_1^6q_4q_5 - 550800q_1^4q_2q_4q_5 + 97200q_1^2q_2^2q_4q_5 + 97200q_2^3q_4q_5 \\
&\quad -259200q_1^3q_3q_4q_5 + 259200q_1q_2q_3q_4q_5 + 259200q_3^2q_4q_5 - 97200q_1^2q_4^2q_5 + 291600q_2q_4^2q_5 + 82944q_1^5q_5^2 \\
&\quad +414720q_1^3q_2q_5^2 + 311040q_1q_2^2q_5^2 + 414720q_1^2q_3q_5^2 + 207360q_2q_3q_5^2 + 311040q_1q_4q_5^2 + 41472q_5^3 - 23400q_1^9s_6 \\
&\quad +280800q_1^7q_2s_6 - 226800q_1^5q_2^2s_6 - 1620000q_1^3q_2^3s_6 + 243000q_1q_2^4s_6 - 820800q_1^6q_3s_6 + 216000q_1^4q_2q_3s_6 \\
&\quad +3240000q_1^2q_2^2q_3s_6 + 648000q_2^3q_3s_6 - 864000q_1^3q_3^2s_6 + 864000q_1q_2q_3^2s_6 - 576000q_3^3s_6 + 1555200q_1^5q_4s_6 \\
&\quad +2592000q_1^3q_2q_4s_6 - 1944000q_1q_2^2q_4s_6 - 2592000q_2q_3q_4s_6 - 972000q_1q_4^2s_6 - 2073600q_1^4q_5s_6 \\
&\quad -6220800q_1^2q_2q_5s_6 - 1555200q_2^2q_5s_6 - 4147200q_1q_3q_5s_6 - 1555200q_4q_5s_6 + 10368000q_1^3s_6^2 + 15552000q_1q_2s_6^2 \\
&\quad +5184000q_3s_6^2).
\end{aligned}
$$

The reflection R_2 acts on $(x_1,x_2,x_3,x_4,x_5,x_6)$ by

$$R_2 : x_j \mapsto x_j - \frac{1}{3}\sum_{k=1}^{6} x_k \quad (j = 1,2,\cdots,6).$$

If $f = f(x_1,\cdots,x_6)$ is a polynomial of $(x_1,\cdots,x_6)$, we write $f \circ R_2$ for $f(R_2(x_1,\cdots,x_6))$.

Then by a direct computation, we have

$$
\begin{array}{rcl}
p_1 \circ R_2 & = & \frac{1}{9}(q_1^3 - 9q_1q_2 + 9q_3), \\
p_2 \circ R_2 & = & \frac{1}{9720}(-79q_1^6 + 585q_1^4q_2 + 3645q_1^2q_2^2 - 1215q_2^3 - 3960q_1^3q_3 - 9720q_1q_2q_3 + 3240q_3^2 + 8910q_1^2q_4 + 7290q_2q_4 \\
& & -7776q_1q_5 - 58320s_6), \\
p_3 \circ R_2 & = & \frac{1}{524880}(-1384q_1^9 + 22707q_1^7q_2 - 85293q_1^5q_2^2 - 91125q_1^3q_2^3 + 98415q_1q_2^4 - 57375q_1^6q_3 + 441045q_1^4q_2q_3 \\
& & +338985q_1^2q_2^2q_3 - 98415q_2^3q_3 - 456840q_1^3q_3^2 - 262440q_1q_2q_3^2 + 58320q_3^3 + 82134q_1^5q_4 - 634230q_1^3q_2q_4 \\
& & -393660q_1q_2^2q_4 + 1027890q_1^2q_3q_4 + 196830q_2q_3q_4 - 393660q_1q_4^2 - 53946q_1^4q_5 + 446148q_1^2q_2q_5 + 118098q_2^2q_5 \\
& & -524880q_1q_3q_5 + 236196q_4q_5 + 612360q_1^3s_6 + 3936600q_1q_2s_6 - 1574640q_3s_6), \\
p_4 \circ R_2 & = & \frac{1}{283435200}(-61441q_1^{12} + 1761678q_1^{10}q_2 - 17211690q_1^8q_2^2 + 57445200q_1^6q_2^3 + 2460375q_1^4q_2^4 - 32476950q_1^2q_2^5 \\
& & -2696580q_1^9q_3 + 59428080q_1^7q_2q_3 - 327612600q_1^5q_2^2q_3 - 23619600q_1^3q_2^3q_3 + 64953900q_1q_2^4q_3 - 34739280q_1^6q_3^2 \\
& & +553748400q_1^4q_2q_3^2 + 81356400q_1^2q_2^2q_3^2 - 23619600q_2^3q_3^2 - 235612800q_1^3q_3^3 - 62985600q_1q_2q_3^3 + 10497600q_3^4 \\
& & +1843155q_1^8q_4 - 58480380q_1^6q_2q_4 + 388739250q_1^4q_2^2q_4 + 53144100q_1^2q_2^3q_4 - 13286025q_2^4q_4 + 19595520q_1^5q_3q_4 \\
& & -1165233600q_1^3q_2q_3q_4 - 141717600q_1q_2^2q_3q_4 + 530128800q_1^2q_3^2q_4 + 47239200q_2q_3^2q_4 + 43302600q_1^4q_4^2 \\
& & +543250800q_1^2q_2q_4^2 + 53144100q_2^2q_4^2 - 259815600q_1q_3q_4^2 + 17714700q_4^3 + 93312q_1^7q_5 + 23514624q_1^5q_2q_5 \\
& & -226748160q_1^3q_2^2q_5 + 29393280q_1^4q_3q_5 + 579467520q_1^2q_2q_3q_5 - 201553920q_1q_3^2q_5 - 113374080q_1^3q_4q_5 \\
& & -453496320q_1q_2q_4q_5 + 113374080q_3q_4q_5 + 52907904q_1^2q_5^2 + 68024448q_2q_5^2 + 75232800q_1^6s_6 \\
& & -692841600q_1^4q_2s_6 - 1464415200q_1^2q_2^2s_6 + 1700611200q_1^3q_3s_6 + 2078524800q_1q_2q_3s_6 - 377913600q_3^2s_6 \\
& & -2078524800q_1^2q_4s_6 - 850305600q_2q_4s_6 + 1133740800q_1q_5s_6 + 1700611200s_6^2), \\
p_5 \circ R_2 & = & \frac{1}{61222003200}(190052q_1^{15} + 7543305q_1^{13}q_2 - 314760330q_1^{11}q_2^2 + 3315388725q_1^9q_2^3 - 11699575200q_1^7q_2^4 \\
& & +4687014375q_1^5q_2^5 + 3587226750q_1^3q_2^6 - 597871125q_1q_2^7 + 41009175q_1^{12}q_3 + 135576990q_1^{10}q_2q_3 \\
& & -12521395125q_1^8q_2^2q_3 + 74175385500q_1^6q_2^3q_3 - 20512146375q_1^4q_2^4q_3 - 12356003250q_1^2q_2^5q_3 + 597871125q_2^6q_3 \\
& & +1263778200q_1^9q_3^2 + 7463793600q_1^7q_2q_3^2 - 163560481200q_1^5q_2^2q_3^2 + 17950896000q_1^3q_2^3q_3^2 + 10097379000q_1q_2^4q_3^2 \\
& & +6796029600q_1^6q_3^3 + 138332124000q_1^4q_2q_3^3 + 3070548000q_1^2q_2^2q_3^3 - 2125764000q_2^3q_3^3 - 36321696000q_1^3q_3^4 \\
& & -5668704000q_1q_2q_3^4 + 755827200q_3^5 - 124608780q_1^{11}q_4 + 1213748550q_1^9q_2q_4 + 7755102000q_1^7q_2^2q_4 \\
& & -79344141300q_1^5q_2^3q_4 + 12488863500q_1^3q_2^4q_4 + 8370195750q_1q_2^5q_4 - 6226279650q_1^8q_3q_4 + 17065161000q_1^6q_2q_3q_4 \\
& & +320016055500q_1^4q_2^2q_3q_4 - 2657205000q_1^2q_2^3q_3q_4 - 5978711250q_2^4q_3q_4 - 55642528800q_1^5q_3^2q_4 \\
& & -327840048000q_1^3q_2q_3^2q_4 - 10628820000q_1q_2^2q_3^2q_4 + 81723816000q_1^2q_3^3q_4 + 4251528000q_2q_3^3q_4 \\
& & +5558479200q_1^7q_4^2 - 30711384900q_1^5q_2q_4^2 - 150397803000q_1^3q_2^2q_4^2 - 16740391500q_1q_2^3q_4^2 + 90020200500q_1^4q_3q_4^2 \\
& & +214170723000q_1^2q_2q_3q_4^2 + 7174453500q_2^2q_3q_4^2 - 44641044000q_1q_3^2q_4^2 - 35783694000q_1^3q_4^3 \\
& & -33480783000q_1q_2q_4^3 + 4782969000q_3q_4^3 + 122974524q_1^{10}q_5 - 1648516860q_1^8q_2q_5 + 569232360q_1^6q_2^2q_5 \\
& & +41133533400q_1^4q_2^3q_5 - 11319693300q_1^2q_2^4q_5 - 1434890700q_2^5q_5 + 5427259200q_1^7q_3q_5 - 30377954880q_1^5q_2q_3q_5 \\
& & -148803480000q_1^3q_2^2q_3q_5 + 12754584000q_1q_2^3q_3q_5 + 45381124800q_1^4q_3^2q_5 + 125278358400q_1^2q_2q_3^2q_5 \\
& & -2550916800q_2^2q_3^2q_5 - 24942297600q_1q_3^3q_5 - 8299927440q_1^6q_4q_5 + 58741945200q_1^4q_2q_4q_5 \\
& & +130734486000q_1^2q_2^2q_4q_5 + 5739562800q_2^3q_4q_5 - 118475913600q_1^3q_3q_4q_5 - 127545840000q_1q_2q_3q_4q_5 \\
& & +15305500800q_3^2q_4q_5 + 56757898800q_1^2q_4^2q_5 + 17218688400q_2q_4^2q_5 + 2781444096q_1^5q_5^2 - 23128312320q_1^3q_2q_5^2 \\
& & -24488801280q_1q_2^2q_5^2 + 34012224000q_1^2q_3q_5^2 + 12244400640q_2q_3q_5^2 - 24488801280q_1q_4q_5^2 + 2448880128q_5^3 \\
& & +1452897000q_1^9s_6 - 35980524000q_1^7q_2s_6 + 213119650800q_1^5q_2^2s_6 + 172186884000q_1^3q_2^3s_6 \\
& & -52612659000q_1q_2^4s_6 + 45885009600q_1^6q_3s_6 - 807317928000q_1^4q_2q_3s_6 - 522937944000q_1^2q_2^2q_3s_6 \\
& & +38263752000q_2^3q_3s_6 + 530968608000q_1^3q_3^2s_6 + 289103904000q_1q_2q_3^2s_6 - 34012224000q_3^3s_6 \\
& & -20596291200q_1^5q_4s_6 + 926833104000q_1^3q_2q_4s_6 + 420901272000q_1q_2^2q_4s_6 - 833299488000q_1^2q_3q_4s_6 \\
& & -153055008000q_2q_3q_4s_6 + 210450636000q_1q_4^2s_6 - 11337408000q_1^4q_5s_6 - 510183360000q_1^2q_2q_5s_6 \\
& & -91833004800q_2^2q_5s_6 + 326517350400q_1q_3q_5s_6 - 91833004800q_4q_5s_6 - 578207808000q_1^3s_6^2 \\
& & -1224440064000q_1q_2s_6^2 + 306110016000q_3s_6^2), \\
s_6 \circ R_2 & = & (-13q_1^6 + 180q_1^4q_2 - 405q_1^2q_2^2 - 450q_1^3q_3 + 810q_1q_2q_3 + 810q_1^2q_4 - 972q_1q_5 + 14580s_6)/14580.
\end{array}
$$

We now take one of $m_1, m_2, m_3, m_4, m_5, m_7$ and write it, say m_j. It is possible to show the expressions of both m_j and $m_j \circ R_2$ as polynomials of q_1, q_2, q_3, q_4, q_5 and s_6. Comparing these two expressions, we find that $m_j \circ R_2 = m_j$. This completes the proof of the theorem.

Remark 1 *The normalization of the choice of m_j's is that as a polynomial of x, each m_j contains the term x_1^{6j} with coefficient* 1. *This implies that as a polynomial of $p_3, p_6, p_9, p_{12}, p_{15}, s_6$, the value of m_j at $p_3 = p_6 = p_9 = p_{12} = p_{15} = 1, s_6 = 0$ is* 1.

Corollary 1 *The relations between* $\mu_6, \mu_{12}, \cdots$ *and* $m_1, m_2, \cdots$ *are given by*

$$\begin{aligned}
\mu_6 &= -1944m_1, \\
\mu_{12} &= 66096m_2, \\
\mu_{18} &= -1770984m_3, \\
\mu_{24} &= 47830176m_4, \\
\mu_{30} &= -1291401144m_5, \\
\mu_{42} &= -941431787784m_7.
\end{aligned}$$

Proof. The polynomials m_j ($j = 1, 2, 3, 4, 5, 7$) are obtained by the same idea as the construction of μ_{6j} ($j = 1, 2, 3, 4, 5, 7$) given in [5]. Actually m_j coincides with μ_{6j} up to a constant factor. The constants are specified by evaluating polynomials at special values and the result follows.

4 Two results on invariants and a discriminant of ST34

We give two results which play important roles in our consideration.

4.1 Invariants by Terao and Enta

We recall the invariants $f_1, f_2, f_3, f_4, f_5, f_6$ given in Terao and Enta [15] (see also [11]). By a direct computation we have

$$\begin{aligned}
f_1 &= \tfrac{1}{1944}\mu_6(= -m_1), \\
f_2 &= \tfrac{1}{3888}\mu_{12}(= 17m_2), \\
f_3 &= \tfrac{1}{1944}\mu_{18}(= -911m_3), \\
f_4 &= \tfrac{1937160963}{17975495468750}m_1^4 - \tfrac{31670896436}{19773045015625}m_1^2m_2 + \tfrac{233872961}{754856437500}m_2^2 + \tfrac{63038467}{258156875724}m_1m_3 - \tfrac{6151}{205320951}m_4, \\
f_5 &= \tfrac{1}{1944}\mu_{30}(= -664301m_5), \\
f_6 &= \tfrac{1}{1944}\mu_{42}(= -484275611m_7),
\end{aligned}$$

where f_4 is the determinant of the Hessian matrix of f_1 up to a constant factor.

4.2 Discriminant by Bessis and Michel

Concerning the discriminant of ST34, there are studies by Terao and Enta [15] (see also [11]), Bessis and Michel (unpublished). Bessis and Michel [1] constructed Saito matrices of some complex reflection groups. They commented that the case ST34 is not treated in [1] since the Saito matrix of ST34 is "too large to be printed" (p.261 of [1]). Moreover based on the idea of [15], they constructed the Saito matrix of the invariants by Conway and Sloane explicitly (unpublished).

Prof. J. Michel kindly sent the data of the Saito matrix of ST34 for the basic invariants $\{f_1, f_2, \cdots, f_6\}$ on the request of the second author (J. S.). We call this matrix M_{34} whose determinant is the discriminant of ST34 up to a non-zero constant factor. Each matrix entry of M_{34} is a polynomial of x, y, z, t, u, v which are the same as $f_1, f_2, f_3, f_4, f_5, f_6$ of Terao and Enta , respectively.

5 Construction of the Saito matrix in terms of a potential vector field

The Saito matrix of a reflection group depends on the choice of basic invariants. There is an alternative way to construct the Saito matrix of ST34. We begin this section with explaining the construction of the Saito matrix for the flat coordinate. For the details, see [12], [7], [13].

The Saito matrix of ST34 can be expressed by a potential vector field given in [7]. To show the result, we define the following polynomials h_j $(j = 1, 2, 3, 4, 5, 6)$ of $u_1, u_2, u_3, u_4, u_5, u_6$.

$$
\begin{aligned}
h_1 = {} & (20u_1^4u_2^2 + 120u_1^2u_2^3 - 60u_2^4 - 12u_1^5u_3 + 60u_1^3u_2u_3 - 180u_1u_2^2u_3 + 135u_1^2u_3^2 + 135u_2u_3^2 + 120u_1^4u_4 \\
& -180u_1^2u_2u_4 + 540u_2^2u_4 + 270u_1u_3u_4 + 405u_4^2 + 180u_1^3u_5 + 540u_1u_2u_5 + 405u_3u_5 + 405u_1u_6)/405, \\
h_2 = {} & (64u_1^9 - 288u_1^7u_2 - 1728u_1^5u_2^2 + 1728u_1^3u_2^3 - 2592u_1u_2^4 + 432u_1^6u_3 + 3888u_1^4u_2u_3 + 3888u_1^2u_2^2u_3 \\
& +1944u_2^3u_3 + 972u_1^3u_3^2 + 729u_3^3 - 1296u_1^5u_4 + 7776u_1^3u_2u_4 + 8748u_1^2u_3u_4 - 2916u_2u_3u_4 - 5832u_1u_4^2 \\
& +3888u_1^4u_5 - 2916u_1^2u_2u_5 - 2916u_2^2u_5 + 4374u_1u_3u_5 + 4374u_4u_5 + 4374u_2u_6)/4374, \\
h_3 = {} & (64u_1^{10} + 3456u_1^8u_2 + 6624u_1^6u_2^2 + 2160u_1^4u_2^3 + 3888u_2^5 + 7776u_1^7u_3 + 19440u_1^5u_2u_3 + 32400u_1^3u_2^2u_3 \\
& +7776u_1u_2^3u_3 + 19440u_1^4u_3^2 + 26244u_1^2u_2u_3^2 + 2916u_2^2u_3^2 + 7290u_1u_3^3 + 9504u_1^6u_4 + 38880u_1^4u_2u_4 \\
& -11664u_1^2u_2^2u_4 - 3888u_2^3u_4 + 25272u_1^3u_3u_4 + 34992u_1u_2u_3u_4 + 8748u_3^2u_4 + 5832u_1^2u_4^2 + 8748u_2u_4^2 \\
& +11664u_1^5u_5 + 25272u_1^3u_2u_5 + 17496u_1u_2^2u_5 + 26244u_1^2u_3u_5 + 8748u_2u_3u_5 + 26244u_1u_4u_5 \\
& +6561u_5^2 + 6561u_3u_6)/6561, \\
h_4 = {} & (1152u_1^{11} + 832u_1^9u_2 + 23616u_1^7u_2^2 - 13824u_1^5u_2^3 + 34560u_1^3u_2^4 + 7776u_1^8u_3 + 42768u_1^6u_2u_3 + 16200u_1^4u_2^2u_3 \\
& +7776u_1^2u_2^3u_3 - 5832u_2^4u_3 + 17496u_1^5u_3^2 + 27216u_1^3u_2u_3^2 + 23328u_1u_2^2u_3^2 + 13122u_1^2u_3^3 + 2916u_2u_3^3 \\
& +12960u_1^7u_4 - 25920u_1^5u_2u_4 + 103680u_1^3u_2^2u_4 - 31104u_1u_2^3u_4 + 19440u_1^4u_3u_4 + 69984u_1^2u_2u_3u_4 \\
& -5832u_2^2u_3u_4 + 17496u_1u_3^2u_4 + 38880u_1^3u_4^2 - 46656u_1u_2u_4^2 - 4374u_3u_4^2 + 2592u_1^6u_5 + 21384u_1^4u_2u_5 \\
& -5832u_1^2u_2^2u_5 + 11664u_2^3u_5 + 26244u_1^3u_3u_5 + 17496u_1u_2u_3u_5 + 6561u_3^2u_5 - 8748u_1^2u_4u_5 + 17496u_2u_4u_5 \\
& +13122u_1u_5^2 + 13122u_4u_6)/13122, \\
h_5 = {} & (10496u_1^{12} + 70656u_1^{10}u_2 + 86976u_1^8u_2^2 + 233856u_1^6u_2^3 - 25920u_1^4u_2^4 - 20736u_2^6 + 71808u_1^9u_3 \\
& +264384u_1^7u_2u_3 + 393984u_1^5u_2^2u_3 + 129600u_1^3u_2^3u_3 + 93312u_1u_2^4u_3 + 165888u_1^6u_3^2 + 408240u_1^4u_2u_3^2 \\
& +221616u_1^2u_2^2u_3^2 + 134136u_1^3u_3^3 + 87480u_1u_2u_3^3 + 10935u_3^4 + 22464u_1^8u_4 + 209088u_1^6u_2u_4 + 38880u_1^4u_2^2u_4 \\
& +233280u_1^2u_2^3u_4 + 23328u_2^4u_4 + 241056u_1^5u_3u_4 + 272160u_1^3u_2u_3u_4 + 139968u_1u_2^2u_3u_4 + 209952u_1^2u_3^2u_4 \\
& +87480u_2u_3^2u_4 + 19440u_1^4u_4^2 + 349920u_1^2u_2u_4^2 - 69984u_2^2u_4^2 + 139968u_1u_3u_4^2 - 17496u_4^3 + 10368u_1^7u_5 \\
& +38880u_1^5u_2u_5 + 93312u_1^3u_2^2u_5 - 23328u_1u_2^3u_5 + 134136u_1^4u_3u_5 + 227448u_1^2u_2u_3u_5 + 52488u_2^2u_3u_5 \\
& +104976u_1u_3^2u_5 + 198288u_1^3u_4u_5 + 104976u_1u_2u_4u_5 + 78732u_3u_4u_5 + 78732u_1^2u_5^2 + 26244u_2u_5^2 \\
& +39366u_5u_6)/39366, \\
h_6 = {} & (109056u_1^{14} + 433664u_1^{12}u_2 + 1983744u_1^{10}u_2^2 - 400512u_1^8u_2^3 + 2784768u_1^6u_2^4 - 282240u_1^4u_2^5 \\
& +967680u_1^2u_2^6 + 207360u_2^7 + 403200u_1^{11}u_3 + 2395008u_1^9u_2u_3 + 3709440u_1^7u_2^2u_3 + 6096384u_1^5u_2^3u_3 \\
& -846720u_1^3u_2^4u_3 - 725760u_1u_2^5u_3 + 1611792u_1^8u_3^2 + 4445280u_1^6u_2u_3^2 + 3492720u_1^4u_2^2u_3^2 + 1360800u_1^2u_2^3u_3^2 \\
& +462672u_2^4u_3^2 + 1496880u_1^5u_3^3 + 2857680u_1^3u_2u_3^3 + 734832u_1u_2^2u_3^3 + 489888u_1^2u_3^4 + 91854u_2u_3^4 \\
& +177408u_1^{10}u_4 + 80640u_1^8u_2u_4 + 3499776u_1^6u_2^2u_4 - 3870720u_1^4u_2^3u_4 + 2540160u_1^2u_2^4u_4 - 653184u_2^5u_4 \\
& +1439424u_1^7u_3u_4 + 7039872u_1^5u_2u_3u_4 + 3991680u_1^3u_2^2u_3u_4 + 1741824u_1u_2^3u_3u_4 + 2721600u_1^4u_3^2u_4 \\
& +1388016u_1^2u_2u_3^2u_4 + 244944u_2^2u_3^2u_4 + 857304u_1u_3^3u_4 + 1874880u_1^6u_4^2 - 3175200u_1^4u_2u_4^2 \\
& +5225472u_1^2u_2^2u_4^2 + 925344u_1^3u_3u_4^2 + 2939328u_1u_2u_3u_4^2 + 306180u_3^2u_4^2 + 1469664u_1^2u_4^3 - 816480u_2u_4^3 \\
& -8064u_1^9u_5 + 254016u_1^7u_2u_5 + 217728u_1^5u_2^2u_5 + 362880u_1^3u_2^3u_5 + 544320u_1u_2^4u_5 + 707616u_1^6u_3u_5 \\
& +1632960u_1^4u_2u_3u_5 + 2122848u_1^2u_2^2u_3u_5 - 326592u_2^3u_3u_5 + 1714608u_1^3u_3^2u_5 + 1224720u_1u_2u_3^2u_5 \\
& +183708u_3^3u_5 + 816480u_1^5u_4u_5 + 4572288u_1^3u_2u_4u_5 - 1959552u_1u_2^2u_4u_5 + 2694384u_1^2u_3u_4u_5 \\
& +979776u_2u_3u_4u_5 - 244944u_1u_4^2u_5 + 1102248u_1^4u_5^2 + 979776u_1^2u_2u_5^2 + 489888u_2^2u_5^2 + 734832u_1u_3u_5^2 \\
& +367416u_4u_5^2 + 137781u_6^2)/275562.
\end{aligned}
$$

Let $w_j = j/7$ for $j = 1, 2, 3, 4, 5$ and $w_6 = 1$. Then each h_j is a weighted homogeneous polynomial of $u_1, \ldots, u_6$ if w_k is the weight of u_k $(k = 1, 2, \cdots, 6)$. We shall define 6×6 matrices C and T in the following way. The matrix $C = (C_{ij})$ is defined by

$C_{ij} = \partial_{u_i} h_j$ $(i, j = 1, 2, \cdots 6)$. At this moment, we remark one of the important properties of C. Namely the matrices $\partial_{u_j} C$ $(j = 1, \ldots, 6)$ are commutative each other. For $E = \sum_{j=1}^{6} w_j u_j \partial_{u_j}$, we set $T = EC$. Then T is regarded as a Saito matrix of the polynomial $F = \det(T)$, where F is a polynomial of $u_1, u_2, u_3, u_4, u_5, u_6$. It is expected that T is the Saito matrix of ST34 if we regard $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ as a set of appropriate basic invariants of ST34. Actually this is the case as is shown in the theorem below:

Theorem 2 *We regard $u_1, u_2, \cdots, u_6$ as basic invariants of* ST34 *which are polynomials of $f_1, f_2, \cdots, f_6$ defined by the following identities:*

$$\left\{\begin{array}{lcl} f_1 & = & k_1 u_1, \\ f_2 & = & 3k_1^2(484u_1^2 + 125u_2)/56, \\ f_3 & = & 3k_1^3(3840232u_1^3 + 1647300u_1u_2 + 466785u_3)/3136, \\ f_4 & = & -k_1^4(10u_1^2u_2 - 6u_2^2 - 3u_1u_3 + 3u_4)/175616, \\ f_5 & = & 27k_1^5(8037231129928u_1^5 + 7056011495960u_1^3u_2 + 923992672120u_1u_2^2 + 4020065168070u_1^2u_3 \\ & & +320350835370u_2u_3 + 1029935376420u_1u_4 + 170567816805u_5)/9834496, \\ f_6 & = & (41535291386925640512u_1^7 + 55402268263969532360u_1^5u_2 + 17134764422423982880u_1^3u_2^2 \\ & & +1037814693245737680u_1u_2^3 + 38364019383116279580u_1^4u_3 + 13847169906634165920u_1^2u_2u_3 \\ & & +402156952864771860u_2^2u_3 + 1820695718372802075u_1u_3^2 + 14361697034479016040u_1^3u_4 \\ & & +2535507816645416760u_1u_2u_4 + 322231741664151375u_3u_4 + 5379672850162662240u_1^2u_5 \\ & & +306822518227402695u_2u_5 + 60643659460340565u_6)/120472576, \end{array}\right. \tag{1}$$

where $k_1 = (64/27)^{1/7}$. Then $\det(T)$ *is the discriminant of* ST34 *for the basic invariants $u_1, u_2, \cdots, u_6$.*

Proof. We first note that by the identities (1), $\{u_1, u_2, \cdots, u_6\}$ is a set of basic invariants of ST34. Hence we see that $\det(T)$ is an invariant of ST34 of degree 252. Since the action of ST34 on the set of hyperplanes fixed by the pseudo-reflections of ST34 are transitive, $\det(T)$ is irreducible. We denote by ℓ_i $(i = 1, 2, \ldots, 126)$ the linear forms fixed by the pseudo-reflections of ST34. Since the product $\prod_{i=1}^{126} \ell_i^2$ is the discriminant of ST34, the theorem follows if $\det(T)$ is $\prod_i \ell_i^2$ up to a constant factor. Note that $\prod_i \ell_i$ is a relative invariant of ST34 of degree 126. Suppose that $\det(T)$ is divisible by $x_5 - x_6$, which is one of the linear forms ℓ_i. Then $\det(T)/\prod_i \ell_i$ is a relative invariant of degree 126. Because of Corollary 6.43 at p.231 of [11], we conclude that $\det(T)/\prod_i \ell_i$ coincides with $\prod_i \ell_i$ up to a constant factor. This leads to $\det(T) = (\text{constant}) \prod_i \ell_i^2$. Therefore in order to show that $\det(T)$ is $\prod_i \ell_i^2$ up to a constant factor, we only need to show that $\det(T)$ is divisible by $x_5 - x_6$. Moreover, since $\det(T)$ is homogeneous, it is sufficient to show that $\det(T) = 0$ under the condition $x_5 = x_6 = 1$.

To accomplish our purpose, we need some preparation. We define symmetric polynomials r_k in x_1, x_2, x_3, x_4 by

$$r_{3j} = x_1^{3j} + x_2^{3j} + x_3^{3j} + x_4^{3j} \; (j = 1, 2, 3, 4), \; r_4 = x_1 x_2 x_3 x_4,$$

and

$$\tilde{p}_{3j} = p_{3j}|_{x_5 = x_6 = 1} \; (j = 1, 2, 3, 4, 5), \quad \tilde{s}_6 = s_6|_{x_5 = x_6 = 1}.$$

Then

$$\tilde{p}_{3j} = r_{3j} + 2 \; (j = 1, 2, 3), \quad \tilde{s}_6 = r_4,$$

and

$$\begin{aligned}\tilde{p}_{12} &= \tfrac{1}{6}(12 + r_3^4 - 6r_3^2 r_6 + 3r_6^2 + 8r_3 r_9 - 24r_4^3),\\ \tilde{p}_{15} &= \tfrac{1}{6}(12 + r_3^5 - 5r_3^3 r_6 + 5r_3^2 r_9 + 5r_6 r_9 - 30r_3 r_4^3).\end{aligned}$$

Therefore each matrix entry of $\tilde{T} = T|_{x_5=x_6=1}$ is a polynomial of r_{3j} $(j = 1, 2, 3)$ and r_4. Using this expression, we conclude $\det(\tilde{T}) = 0$ by a direct computation with the help of computer softwares.

Remark 2 *Kato, Mano and Sekiguchi [7], [8] formulated a generalization of Frobenius manifold structure and among others they introduced the notion of flat coordinates for well-generated complex reflection groups. As to the group* ST34 *the second author (J.S.) constructed the Saito matrix for the flat coordinate. As a consequence, determinant expression of the discriminant of* ST34 *for the flat coordinate was established (cf. [12]). In the course of the identification of $u_1, u_2, \cdots, u_6$ with basic invariants of* ST34*, we use the Saito matrix constructed by Bessis and Michel. It is worthwhile to mention the procedure of this identification. Noting that the weight of f_j is supposed to be w_j $(= \deg f_i/42)$, we determine the undetermined constants c_{ij} and c_0 by the conditions*

$$\begin{aligned} f_1 &= c_{11}u_1,\\ f_2 &= c_{21}u_2 + c_{22}u_1^2,\\ f_3 &= c_{31}u_3 + c_{32}u_1u_2 + c_{33}u_1^3,\\ f_4 &= c_{41}u_4 + c_{42}u_3u_1 + c_{43}u_2^2 + c_{44}u_2u_1^2 + c_{45}u_1^4,\\ f_5 &= c_{51}u_5 + c_{52}u_4u_1 + c_{53}u_3u_2 + \cdots,\\ f_6 &= c_{61}u_6 + c_{62}u_5u_2 + c_{63}u_5u_1^2 + \cdots \end{aligned}$$

and

$$F = c_0 \det(M_{34}).$$

Solving these equations, we determine the constants c_{ij} and c_0. The answer is given above (1) which already appeared in [12]. It is underlined here that the proof of Theorem 2 is mainly indebted to (1) and is independent of the result of Bessis and Michel on the explicit form of the Saito matrix M_{34}.

6 More about the group ST34

6.1 Minimal vectors of $\Lambda^{(3)}$ and the reflections of ST34

We consider the correspondence between the totality of minimal vectors of $\Lambda^{(3)}$ and that of hyperplanes fixed by pseudo-reflections of ST34. The group generated by P_1, P_2, Q_1, R_1, R_2 is identified with ST33 (the group numbered as 33 in [14]). From Table VII of [14] the order of the center of ST33 is 2. The 9-th power of the matrix $R_2R_1Q_1P_1P_2$ given at p.298 of [14] is

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

which commutes with all generators of ST33. Hence this matrix is a generator of the center of ST33, however, does not commute with the reflection P_3. We have that the center Z_{34} of ST34 centralizes ST33, $Z_{34} \cap \mathrm{ST33} = \{1\}$ and $|Z_{34}| = 6$. As a consequence $H = \mathrm{ST33} \cdot Z_{34}$ is well-defined as a group and $|H/\mathrm{ST33}| = 6$.

There is a natural map between coset spaces

$$\mathrm{ST34}/\mathrm{ST33} \longrightarrow \mathrm{ST34}/H,$$

in other words,

$$756 \text{ minimal vectors of } \Lambda^{(3)} \xrightarrow{6:1} 126 \text{ reflections of ST34.}$$

We shall construct this map concretely. For an element $\frac{1}{\sqrt{3}}(\theta, -\omega^b\theta, 0, 0, 0, 0)$ up to Z_{34}, we assign $x_1 - \omega^b x_2 = 0$ to this vector. We take up $\frac{1}{\sqrt{3}}(1, \omega^b, \omega^c, \omega^d, \omega^e, \omega^f), b+c+d+e+f \equiv 0 \pmod 3$. Without loss of generality, we may assume $0 \le b \le c \le d \le e \le f \le 2$. Under this condition, we solve $b + c + d + e + f \equiv 0 \pmod 3$. First we have a trivial solution $(b, c, d, e, f) = (0, 0, 0, 0, 0)$ which corresponds to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$.

Case $(b, c, d, e, f) = (0, 0, 0, 0, f), f \neq 0$: This has no solution.

Case $(b, c, d, e, f) = (0, 0, 0, e, f), e \neq 0$: If $e = 2$ then $f = 2$ and this is not the case. Let $e = 1$. If $f = 2$ then this is the case corresponding to $x_1+x_2+x_3+x_4+\omega x_5+\omega^2 x_6 = 0$. The case $f = 1$ does not satisfy the condition.

Case $(b, c, d, e, f) = (0, 0, d, e, f), d \neq 0$: If $d = 2$, then $(e, f) = (2, 2)$ satisfies the condition and this corresponds to $x_1 + x_2 + x_3 + \omega^2(x_4 + x_5 + x_6) = 0$. However, this is the same type as the one we shall see below, that is, the case $(b, c, d, e, f) = (0, 0, 1, 1, 1)$. Let $d = 1$. If $e = 2$, then $f = 2$ and this is not the case. Let $e = 1$. Then $f = 2$ is not the case, but the case $f = 1$ works. This corresponds to $x_1 + x_2 + x_3 + \omega(x_4 + x_5 + x_6) = 0$.

Case $(b, c, d, e, f) = (0, c, d, e, f), c \neq 0$: If $c = 2$, then $(d, e, f) = (2, 2, 2)$ and this is not the case. Let $c = 1$. If $d = 2$, then $(e, f) = (2, 2)$ does not satisfy the condition. $(d, e, f) = (1, 2, 2)$ is the case and this corresponds to $x_1+x_2+\omega(x_3+x_3)+\omega^2(x_5+x_6) = 0$. The cases $(e, f) = (1, 2), (1, 1)$ do not satisfy the condition.

Case $(b, c, d, e, f), b \neq 0$: If $b = 2$ then $(c, d, e, f) = (2, 2, 2, 2)$ does not satisfy the condition. Let $b = 1$. If $c = 2$, then $(d, e, f) = (2, 2, 2)$ satisfies the condition and this corresponds to $x_1 + \omega x_2 + \omega^2(x_3 + x_4 + x_5 + x_6) = 0$. However, this type already appeared in Case $(b, c, d, e, f) = (0, 0, 0, 0, 1, 2)$ as type $x_1 + x_2 + x_3 + x_4 + \omega x_5 + \omega^2 x_6 = 0$. Let $c = 1$. If $d = 2$, then $(e, f) = (2, 2)$ does not satisfy the condition. Let $d = 1$. The case $(e, f) = (2, 2)$ does not satisfy the condition. The case $(e, f) = (1, 2)$ satisfies the condition corresponding to $x_1 + \omega(x_2 + x_3 + x_4 + x_5) + \omega^2 x_6 = 0$, but this case already appeared in Case $(b, c, d, e, f) = (0, 0, 0, 1, 2)$ as type $x_1 + x_2 + x_3 + x_4 + \omega x_5 + \omega^2 x_6 = 0$. The case $(e, f) = (1, 1)$ does not satisfy the condition.

We have thus obtained the following proposition.

Proposition 1 *There is a natural 6 to 1 correspondence between the minimal vectors of $\Lambda^{(3)}$ up to Z_{34} and the hyperplanes fixed by the pseudo-reflections of* ST34. *Typical* *correspondences are given by*

$$\begin{aligned}
(\theta, -\omega^b\theta, 0, 0, 0, 0) &\to x_1 - \omega^b x_2 = 0,\ \ b \in \{0,1,2\}, \\
(1,1,1,1,1,1) &\to x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0, \\
(1,1,1,1,\omega,\omega^2) &\to x_1 + x_2 + x_3 + x_4 + \omega x_5 + \omega^2 x_6 = 0, \\
(1,1,1,\omega,\omega,\omega) &\to x_1 + x_2 + x_3 + \omega(x_4 + x_5 + x_6) = 0, \\
(1,1,\omega,\omega,\omega^2,\omega^2) &\to x_1 + x_2 + \omega(x_3 + x_4) + \omega^2(x_5 + x_6) = 0.
\end{aligned}$$

6.2 Basic invariants of ST33 and those of ST34

We now mention the relationship between the basic invariants of the group ST33 and those of ST34. For this purpose, we recall the symmetric polynomials r_k in x_1, x_2, x_3, x_4 introduced in the proof of Theorem 2. If we let $x_5 = x_6 = y$ in the definition of m_j $(j = 1,2,3,4,5,7)$ at §1, then we get polynomials in x_1, x_2, x_3, x_4, y. Since we have

$$p_{3j}|_{x_5=x_6=y} = r_{3j} + 2y^{3j}\ (j = 1,2,3,4,5,6), \quad s_6|_{x_5=x_6=y} = r_4 y^2,$$

we can regard $m_j|_{x_5=x_6=y}$ as a polynomial in r_3, r_4, r_6, r_9, y. This procedure gives us the invariants of ST33. First we set

$$\begin{aligned}
K_4 &= 3r_4 - r_3 y + y^4, \\
K_{10} &= \tfrac{1}{2}(r_3^2 r_4 - 3r_6 r_4 + 2r_3^3 y - 8r_3 r_6 y + 6r_9 y + 36r_4^2 y^2 - 8r_3 r_4 y^3 - 8r_3^2 y^4 + 8r_6 y^4 + 64 r_4 y^6)
\end{aligned}$$

and

$$K_{6k} = m_k|_{x_5=x_6=y}\ (k = 1,2,3,4,5,7).$$

Then we get

$$\begin{aligned}
K_6 &= -5r_3^2 + 6r_6 - 180r_4y^2 - 20r_3y^3 - 8y^6, \\
K_{12} &= \tfrac{1}{34}(475r_3^4 - 540r_3^2r_6 + 2349r_6^2 - 2250r_3r_9 + 81000r_4^3 + 103950r_3^2r_4y^2 - 80190r_6r_4y^2 \\
&\quad +30800r_3^3y^3 - 83160r_3r_6y^3 + 49500r_9y^3 + 1871100r_4^2y^4 + 415800r_3r_4y^5 + 50820r_3^2y^6 \\
&\quad -33264r_6y^6 + 255420r_4y^8 + 6380r_3y^9 + 3676y^{12}), \\
K_{18} &= \tfrac{1}{1822}(-86585r_3^6 + 1859625r_3^4r_6 - 9692055r_3^2r_6^2 + 3982527r_6^3 - 86580r_3^3r_9 \\
&\quad +12427020r_3r_6r_9 - 8402130r_9^2 - 80904420r_3^2r_4^3 + 6342300r_6r_4^3 - 167241240r_3^4r_4y^2 \\
&\quad +452096640r_3^2r_6r_4y^2 - 13384440r_6^2r_4y^2 - 271985040r_3r_9r_4y^2 - 5634849240r_4^4y^2 \\
&\quad -28007160r_3^5y^3 + 97895520r_3^3r_6y^3 - 45110520r_3r_6^2y^3 - 54455760r_3^2r_9y^3 + 29082240r_6r_9y^3 \\
&\quad -5768197920r_3r_4^3y^3 - 6947020080r_3^2r_4^2y^4 + 4465941480r_6r_4^2y^4 - 3216213000r_3^3r_4y^5 \\
&\quad +7443235800r_3r_6r_4y^5 - 4300536240r_9r_4y^5 - 603515640r_3^4y^6 + 1734248880r_3^2r_6y^6 \\
&\quad -18044208r_6^2y^6 - 1099360080r_3r_9y^6 - 60989423040r_4^3y^6 - 27788080320r_3r_4^2y^7 \\
&\quad -5927021100r_3^2r_4y^8 + 4572273420r_6r_4y^8 - 1157156000r_3^3y^9 + 3124321200r_3r_6y^9 \\
&\quad -2006741880r_9y^9 - 18856197360r_4^2y^{10} - 2278916640r_3r_4y^{11} \\
&\quad -126792120r_3^2y^{12} + 127646064r_6y^{12} - 200564640r_4y^{14} - 5051040r_3y^{15} - 4823744y^{18}).
\end{aligned}$$

We can show that $K_4, K_6, K_{10}, K_{12}, K_{18}$ generate the invariant ring of ST33. It is underlined here that a set of the generators of the invariant ring of ST33 was determined by Burkhardt [3]. The invariants of Burkardt are also given in [9] or see the Appendix of this paper. Moreover, we can see that K_{24}, K_{30}, K_{42} are polynomials of K_4, K_6, K_{10}, K_{12}, K_{18} as follows.

$$
\begin{aligned}
K_{24} \quad = \quad & \tfrac{1}{362524562500}(3749217437791K_{12}^2 - 72606621297375000K_{10}^2K_4 - 548583360913500K_{12}K_4^3 \\
& +24202207099125000K_4^6 + 2954928140625K_{18}K_6 - 12101103549562500K_{10}K_4^2K_6 - 19382588618832K_{12}K_6^2 \\
& +548583360913500K_4^3K_6^2 + 13040967602916K_6^4), \\
K_{30} \quad = \quad & \tfrac{1}{86134928412500}(398092491836609625000K_{10}^3 + 1638875315504600K_{12}K_{18} - 902342981496315150 0K_{10}K_{12}K_4^2 \\
& -12948940092446250 0K_{18}K_4^3 + 79618498367321925000 0K_{10}K_4^5 - 3829737121534827K_{12}^2K_6 \\
& -1003393893372626250 00K_{10}^2K_4K_6 + 25366379163880500K_{12}K_4^3K_6 + 3344646311242087500 0K_4^6K_6 \\
& +1580814161617275K_{18}K_6^2 - 7699801741247286000K_{10}K_4^2K_6^2 - 10484943427856196K_{12}K_6^3 \\
& +10412302176058200 0K_4^3K_6^3 + 11181126000681648K_6^5), \\
K_{42} \quad = \quad & \tfrac{1}{7401652548230514062500}(41739180083362459537482431250 0K_{10}^3K_{12} + 1797658480288030291174075K_{12}^2K_{18} \\
& -15459093584438022965939062500K_{10}^2K_{18}K_4 - 9460880818895490828496017750K_{10}K_{12}^2K_4^2 \\
& +2224501683172245967284506250000 0K_{10}^3K_4^3 - 25256901756063221659038750 0K_{12}K_{18}K_4^3 \\
& +133900398318629161000080337500 0K_{10}K_{12}K_4^5 + 877090136589414308560781250 0K_{18}K_4^6 \\
& -4449003366344491934569012500000 0K_{10}K_4^8 - 6443664189964048153587159K_{12}^3K_6 \\
& +5043846719660731057031 25K_{18}^2K_6 - 27215413015553817885770625 00K_{10}^2K_{12}K_4K_6 \\
& -2576515597406337160989843750K_{10}K_{18}K_4^2K_6 + 8009064415023676675114972 50K_{12}^2K_4^3K_6 \\
& +55612542079306149182112656250 00K_{10}^2K_4^4K_6 - 2098303889942454234118706250 0K_{12}K_4^6K_6 \\
& +127953062838631494268332412500 0K_{10}^3K_6^2 - 370850790506796659157525K_{12}K_{18}K_6^2 \\
& -19995403641453544837088839500K_{10}K_{12}K_4^2K_6^2 - 2993968528862776452557062 50K_{18}K_4^3K_6^2 \\
& +205484087525358746611549350000 0K_{10}K_4^5K_6^2 - 142379029607508972386290 98K_{12}^2K_6^3 \\
& -693238804399644118392026250 00K_{10}^2K_4K_6^3 + 1329325545599266972946080500K_{12}K_4^3K_6^3 \\
& +4138030930884967145348550000 0K_4^6K_6^3 + 63445090000090696420950K_{18}K_6^4 \\
& +1744871417009573672762157600 0K_{10}K_4^2K_6^4 + 302087101604345357238915 48K_{12}K_6^5 \\
& -15782661166547247786114840 00K_4^3K_6^5 - 1151437880891875725175341 6K_6^7).
\end{aligned}
$$

Appendix

In this Appendix, we give the explicit relationship between the invariants of Burkhardt and our invariants at §6.2. We note that the variables used by Burkhardt are Y_0, Y_1, Y_2, Y_3, Y_4 which relate with our variables by $x_j = -2Y_j$ $(j = 1, 2, 3, 4)$ and $y = Y_0$. Burkhardt invariants $J_j(Y_0, Y_1, Y_2, Y_3, Y_4)$ $(j = 4, 6, 10, 12, 18)$ can be expressed by our K_j's as

$$
\begin{aligned}
J_4(Y_0, Y_1, Y_2, Y_3, Y_4) &= K_4, \\
J_6(Y_0, Y_1, Y_2, Y_3, Y_4) &= -\frac{1}{8}K_6, \\
J_{10}(Y_0, Y_1, Y_2, Y_3, Y_4) &= \frac{1}{512}K_{10}, \\
J_{12}(Y_0, Y_1, Y_2, Y_3, Y_4) &= \frac{1}{384000}(-750K_4^3 - 17K_6^2 + 17K_{12}), \\
J_{18}(Y_0, Y_1, Y_2, Y_3, Y_4) &= \frac{1}{9177366528000}(140035500K_4^2K_{10} - 115027K_6^3 + 137802K_6K_{12} - 22775K_{18}),
\end{aligned}
$$

respectively.

Remark 3 *There are some errors in the expressions of Burkhardt invariants in [3], as pointed out by, for example, Coble [4, p.350], Hunt [6, p.154], Orlik [10, p.224], Nagano-Shiga [9, Remark 5.3]. The readers should be careful if they compare the invariants in [3] with the above expressions of the invariants in terms of* $K_4, K_6, K_{10}, K_{12}, K_{18}$.

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