NEIGHBORS, NEIGHBOR GRAPHS AND INVARIANT RINGS IN CODING THEORY

HIMADRI SHEKHAR CHAKRABORTY*, WILLIAMS CHIARI, TSUYOSHI MIEZAKI, AND MANABU OURA

Abstract. In the present paper, we discuss the class of Type III and Type IV codes from the perspectives of neighbors. Our investigation analogously extends the results originally presented by Dougherty [8] concerning the neighbor graph of binary self-dual codes. Moreover, as an application of neighbors in invariant theory, we show that the ring of the weight enumerators of Type II code d_n^+ and its neighbors in arbitrary genus is finitely generated. Finally, we obtain a minimal set of generators of this ring up to the space of degree 24 and genus 3.

Key Words: Self-dual codes, graphs, neighbor codes, weight enumerators.

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1. Introduction

One of the most celebrated classifications of codes in algebraic coding theory is self-dual codes. The study of this type of codes is immensely significant not only because of its various practical importance, as many of the best-known codes are self-dual, but also its diverse theoretical connections with geometric lattices, block designs and invariant theory. For instance, see [1, 5, 16]. Brualdi and Pless [3] introduced the concept of neighbors, a remarkable notion in the theory of binary self-dual codes. Two binary self-dual codes of length n is known as neighbors if they share a subcode of codimension 1. In a recent study, Dougherty [8] defined neighbor graph of binary self-dual codes, where two codes are connected by an edge if and only if they neighbors.

The main purpose of this paper is to extend the results in [8] to the case of non-binary self-dual codes, namely for Type III and Type IV codes. We define the notion of neighbors for the self-dual codes over any finite field as follows:

*Corresponding author.

Definition 1.1. Two self-dual codes of length n over $\mathbb{F}_q$ are called *neighbors* if they share a subcode of dimension $\frac{n}{2}-1$, that is, their intersection is a subcode of codimension 1.

In this note, we discuss some properties of neighbors in different classes of self-dual codes, specifically, Type III codes, Type III codes containing all-ones vector and Type IV codes. We refer the readers to [4, 13, 18] for a detail discuss on Type III and Type IV codes. We also define the neighbor graphs of above mentioned classes of self-dual codes. We apply this notion in study of Type III and Type IV codes and their neighbors. In particular, we use these graphs to count the number of Type III (resp. Type IV) codes applying the concept of k-neighbors. Moreover, we use the idea of k-neighbors of self-dual codes over finite fields to define the notion of k-neighbor graphs. Using this notion we derive several analogous results of counting formulae.

Definition 1.2. For $0 \leq k \leq \frac{n}{2}$, two self-dual codes of length n over $\mathbb{F}_q$ are called k-neighbors if and only if they share a subcode of dimension $\frac{n}{2}-k$.

Finally, we apply neighbors in invariant theory and prove that the ring of the weight enumerators of Type II code d_n^+ and its neighbors in arbitrary genus can be finitely generated over $\mathbb{C}$. Finally, we show that the space of degree 24 of this ring is strictly smaller than the ring of the weight enumerators of all Type II codes.

This paper is organized as follows. In Section 2, we discuss definitions and the basic properties of linear codes, neighbors and graphs that are needed to understand this paper. In Section 3, we define the neighbor graph of Type III codes, and answer various counting questions in this graph. Using k-neighbors we also derive a new formula to count the number of Type III codes. We illustrate these results in Sections 4 and 5 for other important classes of self-dual codes, for example, Type III codes containing all-ones vector and Type IV codes. In Section 6, we discuss k-neighbor graphs and its properties. Finally, in Section 7, we discuss the invariant ring of the weight enumerators of Type II code d_n^+ and its neighbors.

All computer calculations in this paper were done with the help of Magma [2] and SageMath [22].

2. Preliminaries

In this section, we give a brief discussion on linear codes and graphs including the basic definitions and properties. We follow [12, 14] for the discussions.

2.1. **Linear codes.** Let $\mathbb{F}_q$ be a finite field of order q, where q is a prime power. In this paper, q will be either 2, 3 or 4. Then $\mathbb{F}_q^n$ denotes the vector space of dimension n with inner product:

$$u \cdot v := \begin{cases} u_1v_1 + \cdots + u_nv_n, & \text{if } q = 2, 3 \\ u_1v_1^2 + \cdots + u_nv_n^2, & \text{if } q = 4 \end{cases}$$

for $u, v \in \mathbb{F}_q^n$, where $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$. Here $\mathbb{F}_4 := \{0, 1, \omega, \omega^2\}$ with $1 + \omega + \omega^2 = 0$. We call u and v are *orthogonal* If $u \cdot v = 0$. An element $u \in \mathbb{F}_q^n$ is called *self-orthogonal* if $u \cdot u = 0$. We denote the all-ones vector by $\mathbf{1}$ and zero vector by $\mathbf{0}$. The *weight* $\mathrm{wt}(u)$ of a vector $u \in \mathbb{F}_q^n$ is the number of non-zero coordinates in it. An $\mathbb{F}_q$-*linear code* C of length n is a vector subspace of $\mathbb{F}_q^n$. The elements of C are called *codewords*. The *dual code* of C is defined as

$$C^{\perp} := \{v \in \mathbb{F}_q^n \mid u \cdot v = 0 \text{ for all } u \in C\}.$$

If $C \subseteq C^{\perp}$, then C is called *self-orthogonal*. Clearly, every codeword of a self-orthogonal code is self-orthogonal. In addition, when $C = C^{\perp}$, we call C *self-dual*. It is well known that the length n of a self-dual code over $\mathbb{F}_q$ is even and the dimension is $n/2$.

Lemma 2.1. *Let $n \equiv 0 \pmod 4$. Then the weight of any self-orthogonal vector in $\mathbb{F}_3^n$ is divisible by* 3.

Proof. Since $n \equiv 0 \pmod 4$ and $x^2 = 1$ for any non-zero $x \in \mathbb{F}_3$, therefore any vector in $\mathbb{F}_3^n$ is self-orthogonal if and only if its weight is divisible by 3. □

Lemma 2.2. *Let n is even. Then the weight of each self-orthogonal vector in $\mathbb{F}_4^n$ is even.*

Proof. Since n is even and $x^3 = 1$ for any non-zero $x \in \mathbb{F}_4$, therefore any vector in $\mathbb{F}_4^n$ is self-orthogonal if and only if its weight is even. □

For any self-dual code C of length n over $\mathbb{F}_q$, it is immediate that

$$C_0 := \{w \in C \mid w \cdot v = 0\}$$

is a subcode of C with co-dimension 1, where $v \in \mathbb{F}_q^n$ is a self-orthogonal vector not in C. Then it is not hard to show that $N_C(v) := \langle C_0, v\rangle$ is a neighbor of C, see [8].

In general, a self-dual code C of length n over $\mathbb{F}_q$ has several neighbors. We do not always have that $N_C(v_1) \neq N_C(v_2)$ for $v_1 \neq v_2$. Then by the similar arguments in [8, Lemma 3.2], we have the following useful lemma.

Lemma 2.3. *Let C be a self-dual code of length n over $\mathbb{F}_q$. Let v_1 and v_2 be self-orthogonal vectors in $\mathbb{F}_q^n$ but not in C. Then $N_C(v_1) = N_C(v_2)$ if and only if there exists a vector $w \in C$ such that $w \cdot v_1 = 0$ and $v_2 = w + \alpha v_1$ for any nonzero $\alpha \in \mathbb{F}_q$.*

Lemma 2.4. *Let C be a self-dual code of length n over $\mathbb{F}_q$. Let $v_0 \in \mathbb{F}_q^n$ be a self-orthogonal vector not in C. Then the number of self-orthogonal vectors $v \in \mathbb{F}_q^n$ such that $N_C(v) = N_C(v_0)$ is $(q-1)q^{\frac{n}{2}-1}$.*

2.2. **Graphs.** A *graph* $G := (V, E)$ consists of V, a non-empty set of *vertices*, and E, a set of *edges*. An edge is usually incident with two vertices. But if the edge incident with equal end vertices, the edge is called a *loop*. A graph is called *simple* if it has neither loops nor multiple edges. The *degree* of a vertex v in graph G, denoted by $\deg_G(v)$, is the number of edges that are incident to v. The graph G is called *regular* if $\deg_G(v)$ is same for each vertex v in G. A *path* in G is a sequence of edges $(e_1, e_2, \ldots, e_{m-1})$ having a sequence of vertices $(v_1, v_2, \ldots, v_n)$ satisfying $e_i \mapsto \{v_i, v_{i+1}\}$ for $i = 1, 2, \ldots, m-1$. If there is a path between any two vertices of a graph, then the graph is called *connected*.

3. Type III codes

A self-dual code over $\mathbb{F}_3$ is called *Type* III if the weight of each codeword is congruent to 0 (mod 3). We recall that Type III codes of length n exists if and only if $n \equiv 0 \pmod 4$. Let $T_{\mathrm{III}}(n)$ be the number of Type III codes of length $n \equiv 0 \pmod 4$. Fortunately, we have an explicit formula that gives the number $T_{\mathrm{III}}(n)$ as follows (see [16, 17, 20]):

$$T_{\mathrm{III}}(n) = \prod_{i=0}^{\frac{n}{2}-1} (3^i + 1). \tag{1}$$

Lemma 3.1. *Let $n \equiv 0 \pmod 4$. Then the number of self-orthogonal vectors in $\mathbb{F}_3^n$ is $3^{n-1} + 3^{\frac{n}{2}} - 3^{\frac{n}{2}-1}$.*

Proof. It is immediate from [7, Theorem 65]. However, we give a different proof for this particular case. By Lemma 2.1, the number of self-orthogonal vectors in $\mathbb{F}_3^n$ for $n \equiv 0 \pmod 4$ is

$$C(n, 0) + 2^3 C(n, 3) + 2^6 C(n, 6) + \cdots + 3^n C(n, n),$$

where $C(n, k)$ is the binomial function. Now let ω be the primitive cube root of unity, satisfying $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. Then immediately we can have

$$\begin{aligned}&C(n,0)+2^3C(n,3)+2^6C(n,6)+\cdots+2^nC(n,n)\\&=\frac{3^n+(1+2\omega)^n+(1+2\omega^2)^n}{3}\\&=\frac{3^n+2.3^{\frac{n}{2}}}{3}.\end{aligned}$$

Hence, the number of self-orthogonal vectors is $3^{n-1}+3^{\frac{n}{2}}-3^{\frac{n}{2}-1}$. $\square$

Theorem 3.2. *Let C be a Type* III *code of length $n \equiv 0 \pmod 4$. Then $N_C(v)$ is a Type* III *code if and only if $v \in \mathbb{F}_3^n$ is a self-orthogonal vector not in C.*

Proof. Suppose $N_C(v)$ is a Type III code. Then v must be a self-orthogonal vector, otherwise $N_C(v)$ will no longer a Type III code.

Conversely, suppose that $v \in \mathbb{F}_3^n$ is a self-orthogonal vector not in C. Then by Lemma 2.1, $\mathrm{wt}(v) \equiv 0 \pmod 3$. Since C is a Type III code, therefore $C_0 = \{w \in C \mid w \cdot v = 0\}$ is a subcode of C with co-dimension 1. Let $w \in C_0$. Then $\mathrm{wt}(w+v) \equiv 0 \pmod 3$, since self-orthogonal vectors w and v are orthogonal to each other and

$$(w+v)\cdot(w+v) = w\cdot w + 2w\cdot v + v\cdot v = 0.$$

Therefore, the weight of each vector in $N_C(v)$ is a multiple of 3 and hence $N_C(v)$ is a Type III code. $\square$

Theorem 3.3. *Let $n \equiv 0 \pmod 4$. Let C be a Type* III *code of length n. If $C' = N_C(v)$ for some self-orthogonal vector $v \in \mathbb{F}_3^n$, then $C = N_{C'}(w)$ for some self-orthogonal vector $w \in \mathbb{F}_3^n$.*

Proof. Let C be a Type III code of length $n \equiv 0 \pmod 4$. Let $v \in \mathbb{F}_3^n$ be a self-orthogonal vector not in C. Then by Lemma 2.1, we get $\mathrm{wt}(v) \equiv 0 \pmod 3$. Then $C_0 = \{u \in C \mid u \cdot v = 0\}$ is a subcode of C with co-dimension 1. This implies $C = \langle C_0, w\rangle$ for some self-orthogonal vector w. By Lemma 2.1, $\mathrm{wt}(w) \equiv 0 \pmod 3$. Clearly, w is orthogonal to each vector in C_0. Moreover, by Theorem 3.2, we have $C' = N_C(v)$ is a Type III code. This implies $N_{C'}(w)$ is also a Type III code. Let $C_0' = \{u' \in C' \mid u' \cdot w = 0\}$. Then C_0' is a subcode of C' with co-dimension 1. Since w is orthogonal to each vector in C_0, therefore $C_0' = C_0$. Hence $C = \langle C_0, w\rangle = \langle C_0', w\rangle = N_{C'}(w)$. $\square$

Definition 3.4. Let $n \equiv 0 \pmod 4$. Let $V_{\mathrm{III}}(n)$ be the set of all Type III codes of length n. Let $\Gamma_{\mathrm{III}}(n) := (V_{\mathrm{III}}(n), E_{\mathrm{III}}(n))$ be a graph, where any two vertices in $V_{\mathrm{III}}(n)$ are connected by an edge in $E_{\mathrm{III}}(n)$ if and only if they are neighbors.

The following theorems gives basic properties of $\Gamma_{\mathrm{III}}(n)$ and answers the various counting questions related to it.

Theorem 3.5. *The graph* $\Gamma_{\mathrm{III}}(n)$ *is simple and undirected.*

Proof. Since any Type III code is not a neighbor of itself, therefore $\Gamma_{\mathrm{III}}(n)$ contains no loop. Moreover, by Theorem 3.3, we have if C is connected to C', then C' is connected to C. This implies $\Gamma_{\mathrm{III}}(n)$ is not a directed graph. □

Theorem 3.6. *Let* $n \equiv 0 \pmod 4$*. Then the graph* $\Gamma_{\mathrm{III}}(n)$ *is connected with maximum path length* $\frac{n}{2}$ *between two vertices.*

Proof. Let C_1 and C_2 be two Type III codes of length $n \equiv 0 \pmod 4$. Let $C_2 = \langle v_1, \dots, v_{\frac{n}{2}} \rangle$. Then each v_i is a self-orthogonal vector such that $\mathrm{wt}(v_i) \equiv 0 \pmod 3$. Let $D_1 := N_{C_1}(v_1)$ and $D_i := N_{D_{i-1}}(v_i)$ for $i = 2, \dots, \frac{n}{2}$. Then C_1, D_1, D_2, …, $D_{\frac{n}{2}} = C_2$ is the path from C_1 to C_2. Hence the graph $\Gamma_{\mathrm{III}}(n)$ is connected. By Example 3.7, we can have two Type III codes, say C_1' and C_2' such that the maximum path length between them in $\Gamma_{\mathrm{III}}(4)$ is 2. Now let the following k-times direct sums for positive integer k:

$$C_1 = C_1' \oplus \cdots \oplus C_1',$$
$$C_2 = C_2' \oplus \cdots \oplus C_2'.$$

This implies the length of C_1 and C_2 is $4k$ and $C_1 \cap C_2 = \mathbf{0}$. Hence there exists two Type III codes of length $n \equiv 0 \pmod 4$ such that the maximum path length is $\frac{n}{2}$ in the graph $\Gamma_{\mathrm{III}}(n)$ is $\frac{n}{2}$. □

Example 3.7. Let C_1 be a code of length 4 over $\mathbb{F}_3$ with generator matrix:

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

It is easy to check that C_1 is a Type III code. Then $D_1 := N_{C_1}(v_1)$ is a neighbor of C_1, where $v_1 = (1, 0, 2, 2)$ is a self-orthogonal vector in $\mathbb{F}_3^4$ and not in C_1. Also, $D_2 := N_{D_1}(v_2)$ is a neighbor of D_1, where $v_2 = (0, 1, 2, 1)$ is a self-orthogonal vector in $\mathbb{F}_3^4$ and not in D_1. Immediately, D_1 and D_2 are Type III codes. The generator matrix of D_2 is:

$$\begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

Moreover, we can see that $C_1 \cap D_2 = \mathbf{0}$. This conclude that the maximum path length of the graph $\Gamma_{\mathrm{III}}(4)$ is 2.

Theorem 3.8. *Let* $n \equiv 0 \pmod 4$*. Then the number of vertices in* $\Gamma_{\mathrm{III}}(n)$ *is* $\prod_{i=0}^{\frac{n}{2}-1}(3^i + 1)$*.*

Proof. We can have the number of Type III codes of length $n \equiv 0 \pmod 4$ from (1). This completes the proof. □

Lemma 3.9. *Let $n \equiv 0 \pmod 4$. Let C be a Type* III *code of length n. Suppose $v_0 \in \mathbb{F}_3^n$ be a self-orthogonal vector not in C. Then the number of self-orthogonal vectors $v \in \mathbb{F}_3^n$ such that $N_C(v) = N_C(v_0)$ is $2.3^{\frac{n}{2}-1}$.*

Proof. By Lemmas 2.1 and 2.4, we can obtain the result. □

Theorem 3.10. *Let $n \equiv 0 \pmod 4$. Then the graph $\Gamma_{\mathrm{III}}(n)$ is regular with degree $\frac{1}{2}(3^{\frac{n}{2}} - 1)$.*

Proof. Let C be a Type III code of length $n \equiv 0 \pmod 4$. Then by Lemma 3.1, we have the number of self-orthogonal vectors in $\mathbb{F}_3^n$ but not in C is $3^{n-1} - 3^{\frac{n}{2}-1}$. Moreover, by Lemma 3.9, each Type III code of length n occurs $2.3^{\frac{n}{2}-1}$ times. Hence the degree of each vertex v in $\Gamma_{\mathrm{III}}(n)$ is

$$\deg_{\Gamma_{\mathrm{III}}(n)}(v) = \frac{3^{n-1} - 3^{\frac{n}{2}-1}}{2.3^{\frac{n}{2}-1}} = \frac{3^{\frac{n}{2}} - 1}{2}.$$

□

Theorem 3.11. *Let $n \equiv 0 \pmod 4$. Then the number of edges in $\Gamma_{\mathrm{III}}(n)$ is*

$$\frac{1}{2}\left(\prod_{i=1}^{\frac{n}{2}-1}(3^i + 1)\right)(3^{\frac{n}{2}} - 1).$$

Proof. By Theorem 3.8, the number of vertex in the graph $\Gamma_{\mathrm{III}}(n)$ is $2\prod_{i=1}^{\frac{n}{2}-1}(3^i + 1)$. Since the graph is regular with degree $\frac{1}{2}(3^{\frac{n}{2}} - 1)$. Therefore,

$$2|E_{\mathrm{III}}(n)| = \left(2\prod_{i=1}^{\frac{n}{2}-1}(3^i + 1)\right)\frac{1}{2}(3^{\frac{n}{2}} - 1).$$

This gives the result. □

In the graph $\Gamma_{\mathrm{III}}(n)$, if the shortest path between two vertices has length k, we call the two vertices are in distance k apart. In this case, the corresponding two Type III codes in $V_{\mathrm{III}}(n)$ are called k-neighbors and share a subcode of co-dimension k.

Remark 3.12. Every Type III code in $V_{\mathrm{III}}(n)$ is its 0-neighbor.

Let C be a Type III code of length $n \equiv 0 \pmod 4$. For any non-negative integer k, we denote the number of Type III k-neighbors of C by $L_k^{\mathrm{III}}(n)$ in $\Gamma_{\mathrm{III}}(n)$. By Remark 3.12, we have $L_0^{\mathrm{III}}(n) = 1$. Now the following theorem gives $L_k^{\mathrm{III}}(n)$ for $k > 0$.

Theorem 3.13. *Let $n \equiv 0 \pmod 4$. Let C be a Type* III *code of length n. Then for $k > 0$, we have*

$$L_k^{\text{III}}(n) = \frac{\prod_{i=0}^{k-1}(3^{n-1-i} - 3^{\frac{n}{2}-1})}{\prod_{j=0}^{k-1}(3^{\frac{n}{2}} - 3^{\frac{n}{2}-1-j})}.$$

Proof. Let C be a Type III code of length n. Then

$$L_k^{\text{III}}(n) = \#\{D \in V_{\text{III}}(n) \mid D \text{ is a } k\text{-neighbor of } C\}.$$

Let $S_k(n)$ be the set of k different self-orthogonal vectors $v_1, v_2, \ldots, v_k \in \mathbb{F}_3^n$ and not in C such that each v_j are orthogonal to $v_1, v_2, \ldots, v_{j-1}$. Then

$$L_k^{\text{III}}(n) = \frac{\#S_k(n)}{\#\text{ways each neighbor of } C \text{ is generated}}.$$

By Lemma 3.1, we have the number of self-orthogonal vectors in $\mathbb{F}_3^n$ that are not in C is $3^{n-1} - 3^{\frac{n}{2}-1}$. Moreover, each choice of a self-orthogonal vector v_j reduces the number of available self-orthogonal vectors in ambient space by $\frac{1}{3}$, since it must be orthogonal to the previous v_j and its weight is multiple of 3. This provides that the number of choices for the vectors is $\prod_{i=0}^{k-1}(3^{n-1-i} - 3^{\frac{n}{2}-1})$ the number of choices for self-orthogonal vectors. By using Lemma 2.4 recursively, we can have $\prod_{j=0}^{k-1}(3^{\frac{n}{2}} - 3^{\frac{n}{2}-1-j})$ the number of ways each neighbor of C is generated. Hence

$$L_k^{\text{III}}(n) = \frac{\prod_{i=0}^{k-1}(3^{n-1-i} - 3^{\frac{n}{2}-1})}{\prod_{j=0}^{k-1}(3^{\frac{n}{2}} - 3^{\frac{n}{2}-1-j})}.$$

This completes the proof. □

Remark 3.14. Taking $k = 1$ in the above theorem, we have

$$L_1^{\text{III}}(n) = \frac{3^{n-1} - 3^{\frac{n}{2}-1}}{3^{\frac{n}{2}} - 3^{\frac{n}{2}-1}} = \frac{3^n - 3^{\frac{n}{2}}}{2.3^{\frac{n}{2}}} = \frac{3^{\frac{n}{2}} - 1}{2}.$$

This gives the number of 1-neighbors of C as presented in Theorem 3.10

Example 3.15. Let $n = 4$. Then $T_{\text{III}}(n) = \prod_{i=0}^{\frac{n}{2}-1}(3^i + 1) = 8$. Moreover, $\deg(\Gamma_{\text{III}}(n)) = \frac{1}{2}(3^{\frac{n}{2}} - 1) = 4$. This implies the graph $\Gamma_{\text{III}}(n)$ has 8 vertices and is regular with degree 4. By Remark 3.12, we have $L_k^{\text{III}}(n) = 1$ for $k = 0$. By Theorem 3.13, we have the following k-neighbors of $\Gamma_{\text{III}}(n)$.

For $k = 1$: $L_k^{\text{III}}(n) = \dfrac{3^3 - 3}{3^2 - 3} = 4.$

For $k = 2$: $L_k^{\text{III}}(n) = \dfrac{(3^3 - 3)(3^2 - 3)}{(3^2 - 3)(3^2 - 1)} = 3.$

Then $1 + 4 + 3 = 8$ which is the total number of Type III codes.

The observation in the above example concludes the following result.

Theorem 3.16. *Let* $n \equiv 0 \pmod 4$. *Then the number of Type* III *codes of length* n *is* $\sum_{k=0}^{\frac{n}{2}} L_k^{\mathrm{III}}(n)$.

Proof. Let C_1 and C_2 be any two Type III codes of length $n \equiv 0 \pmod 4$. If C_1 and C_2 are connected by a path in $\Gamma_{\mathrm{III}}(n)$, then by Theorem 3.6, the maximum path length will be $\frac{n}{2}$. This completes the proof. □

Example 3.17. Let $n = 0 \pmod 4$ be the length of the Type III codes, $|V_{\mathrm{III}}(n)|$ the number of vertices in the graphs $\Gamma_{\mathrm{III}}(n)$. In Table 1 we listed the k-neighbors of $\Gamma_{\mathrm{III}}(n)$ up to $n = 12$. Note that in each row k goes from 0 to $\frac{n}{2}$ and sum the sum of the k-neighbors in each row is $|V_{\mathrm{III}}(n)|$ as in Theorem 3.16.

TABLE 1. List of k-Neighbors in $\Gamma_{\mathrm{III}}(n)$ up to $n = 12$

n	$\lvert V_{\mathrm{III}}(n)\rvert$	k-neighbors						
		0	1	2	3	4	5	6
4	8	1	4	3	0	0	0	0
8	2240	1	40	390	1080	729	0	0
12	44817920	1	364	33033	914760	8027019	21493836	14348907

4. TYPE III CODE WITH ALL-ONES VECTOR

In this section, we assume the Type III codes that contain the all-ones vector $\mathbf{1}$. This additional assumption in Type III codes conclude that the length of the code $n \equiv 0 \pmod{12}$.

Lemma 4.1. *Let* $n \equiv 0 \pmod{12}$. *The number of self-orthogonal vectors in* $\mathbb{F}_3^n$ *that are also orthogonal to* $\mathbf{1}$ *is* $3^{n-2} + 3^{\frac{n}{2}} - 3^{\frac{n}{2}-1}$.

Proof. Since $n \equiv 0 \pmod{12}$. By Lemma 3.1, we have the number of self-orthogonal vectors in $\mathbb{F}_3^n$ is $3^{n-1} + 3^{\frac{n}{2}} - 3^{\frac{n}{2}-1}$. Then clearly the number of a self-orthogonal vector that are also orthogonal to $\mathbf{1}$ is $3^{n-2} + 3^{\frac{n}{2}} - 3^{\frac{n}{2}-1}$. □

Theorem 4.2. *Let* $n \equiv 0 \pmod{12}$. *Let* C *be a Type* III *code of length* n *containing all-ones vector. Then* $N_C(v)$ *is a Type* III *code containing all-ones vector if and only if* $v \in \mathbb{F}_3^n$ *is a self-orthogonal vector not in* C *such that* $\mathbf{1} \cdot v = 0$.

Proof. Suppose $N_C(v)$ is a Type III code containing $\mathbf{1}$. Then v must be a self-orthogonal vector such that $\mathbf{1} \cdot v = 0$, otherwise $N_C(v)$ will no longer a Type III code containing $\mathbf{1}$.

Conversely, suppose that $v \in \mathbb{F}_3^n$ is a self-orthogonal vector not in C such that $\mathbf{1} \cdot v = 0$. This implies $v \in \langle \mathbf{1} \rangle^\perp$. Let $C_0 = \{w \in C \mid w \cdot v = 0\}$. Then by Lemma 2.1, $\mathrm{wt}(w + v) \equiv 0 \pmod 3$. Since C is a Type III code, therefore C_0 is a subcode of C with co-dimension 1. This implies that the weight of each vector in $N_C(v)$ is a multiple of 3. Moreover, C contains $\mathbf{1}$ and $\mathbf{1} \cdot v = 0$. Hence $N_C(v)$ is a Type III code containing all-ones vector. □

Theorem 4.3. *Let $n \equiv 0 \pmod{12}$. Let C be a Type* III *code of length n containing all-ones vector. If $C' = N_C(v)$ for some self-orthogonal vector v such that $\mathbf{1} \cdot v = 0$, then $C = N_{C'}(w)$ for some self-orthogonal vector w such that $\mathbf{1} \cdot w = 0$*

Proof. Let C be a Type III code containing $\mathbf{1}$ and v be a self-orthogonal vector not in C such that $\mathbf{1} \cdot v = 0$. This implies $v \in \langle \mathbf{1} \rangle^\perp$. Then $C_0 = \{u \in C \mid u \cdot v = 0\}$ is a subcode of C with co-dimension 1. This implies $C = \langle C_0, w \rangle$ for some self-orthogonal vector w such that $\mathbf{1} \cdot w = 0$. Clearly, w is orthogonal to each vector in C_0. By Lemma 2.1, we have $\mathrm{wt}(v) \equiv 0 \pmod 4$ and $\mathrm{wt}(w) \equiv 0 \pmod 4$. Moreover, by Theorem 4.2, $C' = N_C(v)$ is a Type III code containing $\mathbf{1}$. This implies $N_{C'}(w)$ is also a Type III code. Let $C'_0 = \{u' \in C' \mid u' \cdot w = 0\}$. Then C'_0 is a subcode of C' with co-dimension 1. Since w is orthogonal to each vector in C_0, therefore $C'_0 = C_0$. Hence $C = \langle C_0, w \rangle = \langle C'_0, w \rangle = N_{C'}(w)$. □

Definition 4.4. Let $n \equiv 0 \pmod{12}$. Let $V_{\mathrm{III}}(n, \mathbf{1})$ be the set all Type III codes of length n containing $\mathbf{1}$. Let $\Gamma_{\mathrm{III}}(n, \mathbf{1}) := (V_{\mathrm{III}}(n, \mathbf{1}), E_{\mathrm{III}}(n, \mathbf{1}))$ be a graph, where any two vertices in $V_{\mathrm{III}}(n, \mathbf{1})$ are connected by an edge in $E_{\mathrm{III}}(n, \mathbf{1})$ if and only if they are neighbors.

The following theorems gives basic properties of $\Gamma_{\mathrm{III}}(n)$ and answers the various counting questions related to it.

Theorem 4.5. *Let $n \equiv 0 \pmod{12}$. Then the graph $\Gamma_{\mathrm{III}}(n, \mathbf{1})$ is simple and undirected.*

Proof. Since any Type III code containing $\mathbf{1}$ is not a neighbor of itself, therefore $\Gamma_{\mathrm{III}}(n, \mathbf{1})$ contains no loop. Moreover, by Theorem 4.3, we have if C is connected to C', then C' is connected to C. This implies $\Gamma_{\mathrm{III}}(n, \mathbf{1})$ is not a directed graph. □

Theorem 4.6. *Let $n \equiv 0 \pmod{12}$. The graph $\Gamma_{\mathrm{III}}(n, \mathbf{1})$ is connected with maximum path length $\frac{n}{2} - 1$ between two vertices.*

Proof. Let C_1 and C_2 be two Type III codes of length $n \equiv 0 \pmod{12}$ containing $\mathbf{1}$. Let $C_2 = \langle v_1, \ldots, v_{\frac{n}{2}} \rangle$. Then each v_i is a self-orthogonal vector such that $\mathbf{1} \cdot v_i = \mathbf{0}$. This implies each $\mathbf{v_i} \in \langle \mathbf{1} \rangle^{\perp}$. Let $D_1 = N_{C_1}(v_1)$ and $D_i = N_{D_{i-1}}(v_i)$ for $i = 2, \ldots, \frac{n}{2}$. Then C_1, D_1, D_2, ..., $D_{\frac{n}{2}}$ is the path from C_1 to C_2. Hence the graph $\Gamma_{\mathrm{III}}(n, \mathbf{1})$ is connected. Since each Type III code in $\Gamma_{\mathrm{III}}(n, \mathbf{1})$ contains $\mathbf{1}$, they must have a subspace of dimension 1 in common. This implies that the maximum distance in the graph $\Gamma_{\mathrm{III}}(n, \mathbf{1})$ is $\frac{n}{2} - 1$. □

Theorem 4.7. *Let $n \equiv 0 \pmod{12}$. Then the number of vertices in $\Gamma_{\mathrm{III}}(n, \mathbf{1})$ is $2 \prod_{i=1}^{\frac{n}{2}-2} (3^i + 1)$.*

Proof. We know that the number of Type III codes of length $n \equiv 0 \pmod{12}$ containing $\mathbf{1}$ is $2 \prod_{i=1}^{\frac{n}{2}-2} (3^i + 1)$, see [18]. This completes the proof. □

Lemma 4.8. *Let $n \equiv 0 \pmod{12}$. Let C be a Type III code of length n containing $\mathbf{1}$. Suppose $v_0 \in \mathbb{F}_3^n$ be a self-orthogonal vector not in C such that $\mathbf{1} \cdot v_0 = \mathbf{0}$. Then the number of self-orthogonal vectors $v \in \mathbb{F}_3^n$ such that $N_C(v) = N_C(v_0)$ is $2.3^{\frac{n}{2}-1}$.*

Proof. By Lemmas 2.1 and 2.3, we can obtain the result. □

Theorem 4.9. *Let $n \equiv 0 \pmod{12}$. Then the graph $\Gamma_{\mathrm{III}}(n, \mathbf{1})$ is regular with degree $\dfrac{1}{2}(3^{\frac{n}{2}-1} - 1)$.*

Proof. Let C be a Type III code containing $\mathbf{1}$. Then by Lemma 4.1, we have the number of self-orthogonal vectors in $\mathbb{F}_3^n$ that are orthogonal to $\mathbf{1}$ but not in C is $3^{n-2} - 3^{\frac{n}{2}-1}$. Moreover, by Lemma 4.8, each Type III code of length n occurs $2.3^{\frac{n}{2}-1}$ times. Hence the degree of each vertex v in $\Gamma_{\mathrm{III}}(n, \mathbf{1})$ is

$$\deg_{\Gamma_{\mathrm{III}}(n,\mathbf{1})}(v) = \frac{3^{n-2} - 3^{\frac{n}{2}-1}}{2.3^{\frac{n}{2}-1}} = \frac{3^{\frac{n}{2}-1} - 1}{2}.$$

□

Theorem 4.10. *Let $n \equiv 0 \pmod{12}$. Then the number of edges in the graph $\Gamma_{\mathrm{III}}(n, \mathbf{1})$ is*

$$\frac{1}{2} \left(\prod_{i=1}^{\frac{n}{2}-2} (3^i + 1) \right) (3^{\frac{n}{2}-1} - 1)$$

.

Proof. By Theorem 4.7, the number of vertex in the graph $\Gamma_{\mathrm{III}}(n,\mathbf{1})$ is $2\prod_{i=1}^{\frac{n}{2}-2}(3^i+1)$. Since the graph is regular with degree $\frac{1}{2}(3^{\frac{n}{2}-1}-1)$. Therefore,

$$2|E_{\mathrm{III}}(n,\mathbf{1})| = \left(2\prod_{i=1}^{\frac{n}{2}-2}(3^i+1)\right)\frac{1}{2}(3^{\frac{n}{2}-1}-1).$$

This gives the result. □

Remark 4.11. Every Type III code in $V_{\mathrm{III}}(n,\mathbf{1})$ is its 0-neighbor.

Let C be a Type III code of length $n \equiv 0 \pmod{12}$ containing $\mathbf{1}$. For any non-negative integer k, we denote the number of Type III k-neighbors of C by $L_k^{\mathrm{III}}(n,\mathbf{1})$ in $\Gamma_{\mathrm{III}}(n,\mathbf{1})$. By Remark 4.11, we have $L_0^{\mathrm{III}}(n,\mathbf{1}) = 1$. Now the following theorem gives $L_k^{\mathrm{III}}(n,\mathbf{1})$ for $k>0$.

Theorem 4.12. *Let $n \equiv 0 \pmod{12}$. Let C be a Type* III *code of length n containing $\mathbf{1}$. Then for $k>0$, we have*

$$L_k^{\mathrm{III}}(n,\mathbf{1}) = \frac{\prod_{i=0}^{k-1}(3^{n-2-i}-3^{\frac{n}{2}-1})}{\prod_{j=0}^{k-1}(3^{\frac{n}{2}}-3^{\frac{n}{2}-1-j})}.$$

Proof. Let C be a Type III code of length $n \equiv 0 \pmod{12}$ containing $\mathbf{1}$. Then

$$L_k^{\mathrm{III}}(n,\mathbf{1}) = \#\{D \in V_{\mathrm{III}}(n,\mathbf{1}) \mid D \text{ is a } k\text{-neighbor of } C\}.$$

Let $S_k(n,\mathbf{1})$ be the set of k different self-orthogonal vectors $v_1, v_2, \ldots, v_k \in \mathbb{F}_3^n$ and not in C such that each v_j are orthogonal to $\mathbf{1}, v_1, v_2, \ldots, v_{j-1}$. Then

$$L_k^{\mathrm{III}}(n,\mathbf{1}) = \frac{\#S_k(n,\mathbf{1})}{\#\text{ways each } k\text{-neighbor of } C \text{ is generated}}.$$

By Lemma 4.1, we have the number of self-orthogonal vectors in $\mathbb{F}_3^n$ that are orthogonal to $\mathbf{1}$ and not in C is $3^{n-2}-3^{\frac{n}{2}-1}$. Moreover, each choice of a self-orthogonal vector v_j reduces the number of available self-orthogonal vectors in ambient space by $\frac{1}{3}$, since it must be orthogonal to the previous v_j and its weight is multiple of 3. This provides that the number of choices for the vectors is $\prod_{i=0}^{k-1}(3^{n-2-i}-3^{\frac{n}{2}-1})$ the number of choices for self-orthogonal vectors. By using Lemma 2.4 recursively, we can have $\prod_{j=0}^{k-1}(3^{\frac{n}{2}}-3^{\frac{n}{2}-1-j})$ the number of ways each k-neighbor of C is generated. Hence

$$L_k^{\mathrm{III}}(n,\mathbf{1}) = \frac{\prod_{i=0}^{k-1}(3^{n-2-i}-3^{\frac{n}{2}-1})}{\prod_{j=0}^{k-1}(3^{\frac{n}{2}}-3^{\frac{n}{2}-1-j})}.$$

This completes the proof. □

Remark 4.13. Taking $k = 1$ in the above theorem, we have

$$L_1^{\mathrm{III}}(n, \mathbf{1}) = \frac{3^{n-2} - 3^{\frac{n}{2}-1}}{3^{\frac{n}{2}} - 3^{\frac{n}{2}-1}} = \frac{3^{n-1} - 3^{\frac{n}{2}}}{2.3^{\frac{n}{2}}} = \frac{3^{\frac{n}{2}-1} - 1}{2}.$$

This gives the number of 1-neighbors of a Type III code with $\mathbf{1}$ as presented in Theorem 4.9.

Example 4.14. Let $n = 12$. Then $T_{\mathrm{III}}(n, \mathbf{1}) = \prod_{i=0}^{\frac{n}{2}-2}(3^i + 1) = 183680$. Moreover, $\deg(\Gamma_{\mathrm{III}}(n)) = \frac{1}{2}(3^{\frac{n}{2}-1} - 1) = 121$. This implies the graph $\Gamma_{\mathrm{III}}(n, \mathbf{1})$ has 183680 vertices and is regular with degree 121. By Remark 4.11, we have $L_k^{\mathrm{III}}(n) = 1$ for $k = 0$. By Theorem 4.12, we have the following k-neighbors of $\Gamma_{\mathrm{III}}(n, \mathbf{1})$.

For $k = 1$: $L_k^{\mathrm{III}}(n, \mathbf{1}) = \dfrac{3^{10} - 3^5}{3^6 - 3^5} = 121.$

For $k = 2$: $L_k^{\mathrm{III}}(n, \mathbf{1}) = \dfrac{(3^{10} - 3^5)(3^9 - 3^5)}{(3^6 - 3^5)(3^6 - 3^4)} = 3630.$

For $k = 3$: $L_k^{\mathrm{III}}(n, \mathbf{1}) = \dfrac{(3^{10} - 3^5)(3^9 - 3^5)(3^8 - 3^5)}{(3^6 - 3^5)(3^6 - 3^4)(3^6 - 3^3)} = 32670.$

For $k = 4$: $L_k^{\mathrm{III}}(n, \mathbf{1}) = \dfrac{(3^{10} - 3^5)(3^9 - 3^5)(3^8 - 3^5)(3^7 - 3^5)}{(3^6 - 3^5)(3^6 - 3^4)(3^6 - 3^3)(3^6 - 3^2)} = 88209.$

For $k = 5$: $L_k^{\mathrm{III}}(n, \mathbf{1}) = \dfrac{(3^{10} - 3^5)(3^9 - 3^5)(3^8 - 3^5)(3^7 - 3^5)(3^6 - 3^5)}{(3^6 - 3^5)(3^6 - 3^4)(3^6 - 3^3)(3^6 - 3^2)(3^6 - 3)} = 59049.$

Then $1 + 121 + 3630 + 32670 + 88209 + 59049 = 183680$ which is the total number of Type III codes containing $\mathbf{1}$.

Now we have the following analogous result of Theorem 3.16.

Theorem 4.15. *Let* $n \equiv 0 \pmod{12}$. *The number of Type* III *codes of length* n *containing* $\mathbf{1}$ *is* $\sum_{k=0}^{\frac{n}{2}-1} L_k^{\mathrm{III}}(n, \mathbf{1})$.

Proof. Let C_1 and C_2 be any two Type III codes of length $n \equiv 0 \pmod{12}$ containing $\mathbf{1}$. If C_1 and C_2 are connected by a path in $\Gamma_{\mathrm{III}}(n, \mathbf{1})$, then by Theorem 4.6, the maximum path length will be $\frac{n}{2} - 1$. This completes the proof. □

5. Type IV codes

A self-dual code over $\mathbb{F}_4$ where all codewords have even weights is called *Type* IV code. It is well-known that the Type IV code exists if and only if $n \equiv 0 \pmod 2$. Let $T_{\mathrm{IV}}(n)$ be the number of Type IV codes of length $n \equiv 0 \pmod 2$. From [16, 17, 20], we have an explicit formula that gives the number $T_{\mathrm{IV}}(n)$ as follows :

$$T_{\mathrm{IV}}(n) = \prod_{i=0}^{\frac{n}{2}-1}(2^{2i+1}+1). \tag{2}$$

Lemma 5.1. *Let* $n \equiv 0 \pmod 2$. *Then the number of self-orthogonal vectors in* $\mathbb{F}_4^n$ *is* $2^{n-1}(2^n+1)$.

Proof. By Lemma 2.2, the number of self-orthogonal vectors of length $n \equiv 0 \pmod 2$ is

$$C(n,0) + 3^2C(n,2) + 3^4C(n,4) + \cdots + 3^nC(n,n),$$

where $C(n,k)$ is the binomial function. Then immediately it can be written that

$$\begin{aligned} &C(n,0) + 3^2C(n,2) + 3^4C(n,4) + \cdots + 3^nC(n,n) \\ &= \frac{4^n+2^n}{2} \\ &= 2^{2n-1} + 2^{n-1}. \end{aligned}$$

Hence the number of self-orthogonal vectors is $2^{n-1}(2^n+1)$. □

Theorem 5.2. *Let* $n \equiv 0 \pmod 2$. *Let* C *be a Type* IV *code of length* n. *Then* $N_C(v)$ *is a Type* IV *code if and only if* $v \in \mathbb{F}_4^n$ *is a self-orthogonal vector.*

Proof. Suppose $N_C(v)$ is a Type IV code of length $n \equiv 0 \pmod 2$. Then v must be a self-orthogonal vector with even weight, otherwise $N_C(v)$ will no longer a Type IV code.

Conversely, suppose that $v \in \mathbb{F}_4^n$ is a self-orthogonal vector not in C. Then by Lemma 2.2, $\mathrm{wt}(v) \equiv 0 \pmod 2$. Since C is a Type IV code, therefore $C_0 = \{w \in C \mid w \cdot v = 0\}$ is a subcode of C with co-dimension 1. Let $w \in C_0$. Then $\mathrm{wt}(w+v) \equiv 0 \pmod 2$, since self-orthogonal vectors w and v have even weights and are orthogonal to each other and

$$(w+v)\cdot(w+v) = w\cdot w + 2w\cdot v + v\cdot v = 0.$$

Therefore, the weight of each vector in $N_C(v)$ is even and hence $N_C(v)$ is a Type IV code. □

Theorem 5.3. *Let* $n \equiv 0 \pmod 2$. *Let* C *be a Type* IV *code of length* n. *If* $C' = N_C(v)$ *for some self-orthogonal vector* $v \in \mathbb{F}_4^n$, *then* $C = N_{C'}(w)$ *for some self-orthogonal vector* $w \in \mathbb{F}_4^n$.

Proof. Let C be a Type IV code and v be a self-orthogonal vector not in C. Then by Lemma 2.2, we have $\mathrm{wt}(v)$ is even. Then $C_0 = \{u \in$

$C \mid u \cdot v = 0\}$ is a subcode of C with co-dimension 1. This implies $C = \langle C_0, w\rangle$ for some self-orthogonal vector w. By Lemma 2.2, $\mathrm{wt}(w) \equiv 0 \pmod 2$. Clearly, w is orthogonal to each vector in C_0. Moreover, by Theorem 5.2, $C' = N_C(v)$ is a Type IV code. This implies $N_{C'}(w)$ is also a Type IV code. Let $C_0' = \{u' \in C' \mid u' \cdot w = 0\}$. Then C_0' is a subcode of C' with co-dimension 1. Since w is orthogonal to each vector in C_0, therefore $C_0' = C_0$. Hence $C = \langle C_0, w\rangle = \langle C_0', w\rangle = N_{C'}(w)$. □

Definition 5.4. Let $n \equiv 0 \pmod 2$. Let $V_{\mathrm{IV}}(n)$ be the set all Type IV codes of length n. Let $\Gamma_{\mathrm{IV}}(n) := (V_{\mathrm{IV}}(n), E_{\mathrm{IV}}(n))$ be a graph, where any two vertices in $V_{\mathrm{IV}}(n)$ are connected by an edge in $E_{\mathrm{IV}}(n)$ if and only if they are neighbors.

The following theorems gives basic properties of $\Gamma_{\mathrm{IV}}(n)$ and answers the various counting questions related to it.

Theorem 5.5. *Let $n \equiv 0 \pmod 2$. Then the graph $\Gamma_{\mathrm{IV}}(n)$ is simple and undirected.*

Proof. Since any Type IV code is not a neighbor of itself, therefore $\Gamma_{\mathrm{IV}}(n)$ contains no loop. Moreover, by Theorem 5.3, we have if C is connected to C', then C' is connected to C. This implies $\Gamma_{\mathrm{IV}}(n)$ is not a directed graph. □

Theorem 5.6. *Let $n \equiv 0 \pmod 2$. Then the graph $\Gamma_{\mathrm{IV}}(n)$ is connected with maximum path length $\frac{n}{2}$ between two vertices.*

Proof. Let C_1 and C_2 be two Type IV codes of length $n \equiv 0 \pmod 2$. Let $C_2 = \langle v_1, \ldots, v_{\frac{n}{2}}\rangle$. Then each v_i is a self-orthogonal vector such that $\mathrm{wt}(v_i) \equiv 0 \pmod 2$. Let $D_1 := N_{C_1}(v_1)$ and $D_i := N_{D_{i-1}}(v_i)$ for $i = 2, \ldots, \frac{n}{2}$. Then C_1, D_1, D_2, …, $D_{\frac{n}{2}} = C_2$ is the path from C_1 to C_2. Hence the graph $\Gamma_{\mathrm{IV}}(n)$ is connected. To show the maximum path length between two vertices is $\frac{n}{2}$, let n = 2. Let

$$C_1' = \{(0,0), (1,1), (\omega,\omega), (\omega^2,\omega^2)\}$$

be a code of length 2 over $\mathbb{F}_4$. It is easy to check that C_1' is Type IV. Then immediately we have

$$C_2' = \{(0,0), (1,\omega^2), (\omega,1), (\omega^2,\omega)\},$$

which is a neighbor of C_1' have path length 1. Now let the following k-times direct sums for positive integer k:

$$C_1 = C_1' \oplus \cdots \oplus C_1',$$
$$C_2 = C_2' \oplus \cdots \oplus C_2'.$$

This implies the length of C_1 and C_2 is $2k$ and $C_1 \cap C_2 = \mathbf{0}$. Hence there exists two Type IV codes of length $n \equiv 0 \pmod 2$ such that the maximum path length is $\frac{n}{2}$ in the graph $\Gamma_{\mathrm{IV}}(n)$ is $\frac{n}{2}$. □

Example 5.7. Let C be a code of length 4 over $\mathbb{F}_4$ with generator matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

It is easy to check that C is a Type IV code. Then $D_1 := N_C(v_1)$ is a neighbor of C, where $v_1 = (1, \omega, 0, 0)$ is a self-orthogonal vector in $\mathbb{F}_4^4$ and not in C. Also, $D_2 := N_{D_1}(v_2)$ is a neighbor of D_1, where $v_2 = (0, 0, 1, \omega^2)$ is a self-orthogonal vector in $\mathbb{F}_4^4$ and not in D_1. Immediately, D_1 and D_2 are Type IV codes. The generator matrix of D_2 is:

$$\begin{pmatrix} 1 & \omega & 0 & 0 \\ 0 & 0 & 1 & \omega^2 \end{pmatrix}.$$

Moreover, we can see that $C \cap D_2 = \mathbf{0}$. This conclude that the maximum path length of the graph $\Gamma_{\mathrm{IV}}(4)$ is 2.

Theorem 5.8. *Let $n \equiv 0 \pmod 2$. Then the number of vertices in $\Gamma_{\mathrm{IV}}(n)$ is $\prod_{i=0}^{\frac{n}{2}-1}(2^{2i+1}+1)$.*

Proof. We can have the number of Type IV codes of length $n \equiv 0 \pmod 2$ from (2). This completes the proof. □

Lemma 5.9. *Let $n \equiv 0 \pmod 2$. Let C be a Type IV code of length n. Suppose $v_0 \in \mathbb{F}_4^n$ be a self-orthogonal vector not in C. Then the number of self-orthogonal vectors $v \in \mathbb{F}_4^n$ such that $N_C(v) = N_C(v_0)$ is $3.4^{\frac{n}{2}-1}$.*

Proof. By Lemmas 2.2 and 2.4, we can obtain the result. □

Theorem 5.10. *Let $n \equiv 0 \pmod 2$. Then the graph $\Gamma_{\mathrm{IV}}(n)$ is regular with degree $\frac{2}{3}(2^n - 1)$.*

Proof. Let C be a Type IV code of length $n \equiv 0 \pmod 2$. Then by Lemma 5.1, we have the number of self-orthogonal vectors in $\mathbb{F}_4^n$ but not in C is $2^{2n-1} - 2^{n-1}$. Moreover, by Lemma 5.9, each Type IV code of length n occurs $3.4^{\frac{n}{2}-1}$ times. Hence the degree of each vertex v in $\Gamma_{\mathrm{IV}}(n)$ is

$$\deg_{\Gamma_{\mathrm{IV}(n)}}(v) = \frac{2^{2n-1} - 2^{n-1}}{3.4^{\frac{n}{2}-1}} = \frac{2(2^n - 1)}{3}.$$

□

Theorem 5.11. *Let* $n \equiv 0 \pmod 2$*. Then the number of edge in* $\Gamma_{\mathrm{IV}}(n)$ *is*

$$\left(\prod_{i=1}^{\frac{n}{2}-1}(2^{2i+1}+1)\right)(2^n-1).$$

Proof. By Theorem 5.8, the number of vertex in the graph $\Gamma_{\mathrm{IV}}(n)$ is $2\prod_{i=0}^{\frac{n}{2}-1}(2^{2i+1}+1)$. Since the graph is regular with degree $\frac{2}{3}(2^n-1)$. Therefore,

$$2|E_{\mathrm{IV}}(n)| = \left(3\prod_{i=1}^{\frac{n}{2}-1}(2^{2i+1}+1)\right)\frac{2}{3}(2^n-1).$$

This gives the result. □

In the graph $\Gamma_{\mathrm{IV}}(n)$, if the shortest path between two vertices has length k, we call the two vertices are in distance k apart. In this case, the corresponding two Type IV codes in $V_{\mathrm{IV}}(n)$ are called k-neighbors and share a subcode of co-dimension k.

Remark 5.12. Every Type III code in $V_{\mathrm{IV}}(n)$ is its 0-neighbor.

Let C be a Type IV code of length $n \equiv 0 \pmod 2$. For any non-negative integer k, we denote the number of Type IV k-neighbors of C by $L_k^{\mathrm{IV}}(n)$ in $\Gamma_{\mathrm{IV}}(n)$. By Remark 5.12, we have $L_0^{\mathrm{IV}}(n) = 1$. Now the following theorem gives $L_k^{\mathrm{IV}}(n)$ for $k > 0$.

Theorem 5.13. *Let* $n \equiv 0 \pmod 2$*. Let* C *be a Type* IV *code of length* n*. Then for* $k > 0$*, we have*

$$L_k^{\mathrm{IV}}(n) = \frac{\prod_{i=0}^{k-1}(2^{2n-1-2i}-2^{n-1})}{\prod_{j=0}^{k-1}(2^n-2^{n-2-2j})}.$$

Proof. Let C be a Type IV code of length $n \equiv 0 \pmod 2$. Then

$$L_k^{\mathrm{IV}}(n) = \#\{D \in V_{\mathrm{IV}}(n) \mid D \text{ is a } k\text{-neighbor of } C\}.$$

Let $S_k(n)$ be the set of k different self-orthogonal vectors $v_1, v_2, \ldots, v_k \in \mathbb{F}_4^n$ and not in C such that each v_j are orthogonal to $v_1, v_2, \ldots, v_{j-1}$. Then

$$L_k^{\mathrm{IV}}(n) = \frac{\#S_k(n)}{\#\text{ways each neighbor of } C \text{ is generated}}.$$

By Lemma 5.1, we have the number of self-orthogonal vectors in $\mathbb{F}_4^n$ that are not in C is $2^{2n-1} - 2^{n-1}$.

Moreover, each choice of a self-orthogonal vector v_j reduces the number of available self-orthogonal vectors in ambient space by $\frac{1}{4}$, since it is even weight and must be orthogonal to the previous v_j. This provides that the number of choices for the vectors is $\prod_{i=0}^{k-1}(2^{2n-1-2i} - 2^{n-1})$ the number of choices for self-orthogonal vectors. By using Lemma 2.4 recursively, we can have $\prod_{j=0}^{k-1}(2^n - 2^{n-2-2j})$ the number of ways each neighbor of C is generated. Hence

$$L_k^{\mathrm{IV}}(n) = \frac{\prod_{i=0}^{k-1}(2^{2n-1-2i} - 2^{n-1})}{\prod_{j=0}^{k-1}(2^n - 2^{n-2-2j})}.$$

This completes the proof. □

Remark 5.14. Taking $k = 1$ in the above theorem, we have

$$L_1^{\mathrm{IV}}(n) = \frac{2^{2n-1} - 2^{n-1}}{2^n - 2^{n-2}} = \frac{2^{2n-1} - 2^{n-1}}{3.2^{n-2}} = \frac{2(2^n - 1)}{3}.$$

This gives the number of 1-neighbors of Type IV codes as presented in Theorem 5.10

Example 5.15. Let $n = 6$. Then $T_{\mathrm{IV}}(n) = \prod_{i=0}^{\frac{n}{2}-1}(2^{2i+1} + 1) = 891$. Moreover, $\deg(\Gamma_{\mathrm{IV}}(n)) = \frac{2}{3}(2^n - 1) = 42$. This implies the graph $\Gamma_{\mathrm{IV}}(n)$ has 891 vertices and is regular with degree 42. By Remark 5.12, we have $L_k^{\mathrm{IV}}(n) = 1$ for $k = 0$. By Theorem 5.13, we have the following k-neighbors of $\Gamma_{\mathrm{IV}}(n)$.

For $k = 1$: $L_k^{\mathrm{IV}}(n) = \dfrac{2^{11} - 2^5}{2^6 - 2^4} = 42.$

For $k = 2$: $L_k^{\mathrm{IV}}(n) = \dfrac{(2^{11} - 2^5)(2^9 - 2^5)}{(2^6 - 2^4)(2^6 - 2^2)} = 336.$

For $k = 3$: $L_k^{\mathrm{IV}}(n) = \dfrac{(2^{11} - 2^5)(2^9 - 2^5)(2^7 - 2^5)}{(2^6 - 2^4)(2^6 - 2^2)(2^6 - 1)} = 512$

Then $1 + 42 + 336 + 512 = 891$ which is the total number of Type IV codes.

The following result is the Type IV code analogue of Theorem 3.16.

Theorem 5.16. *Let $n \equiv 0 \pmod 2$. Then the number of Type* IV *codes of length n is $\sum_{k=0}^{\frac{n}{2}} L_k^{\mathrm{IV}}(n)$.*

Proof. Let C_1 and C_2 be any two Type IV codes of length $n \equiv 0 \pmod 4$. If C_1 and C_2 are connected by a path in $\Gamma_{\mathrm{IV}}(n)$, then by Theorem 5.6, the maximum path length will be $\frac{n}{2}$. This completes the proof. □

Example 5.17. Let $n = 0 \pmod 2$ be the length of the Type IV codes, $|V_{\mathrm{IV}}(n)|$ the number of vertices in the graphs $\Gamma_{\mathrm{IV}}(n)$. In Table 2, we listed the k-neighbors of $\Gamma_{\mathrm{IV}}(n)$ up to $n = 10$. Note that in each row k goes from 0 to $\frac{n}{2}$ and sum the sum of the k-neighbors in each row is $|V_{\mathrm{IV}}(n)|$ as in Theorem 5.16.

TABLE 2. List of k-Neighbors in $\Gamma_{\mathrm{IV}}(n)$ up to $n = 10$

n	$\lvert V_{\mathrm{III}}(n)\rvert$	k-neighbors					
		0	1	2	3	4	5
2	3	1	2	0	0	0	0
4	27	1	10	16	0	0	0
6	891	1	42	336	512	0	0
8	114939	1	170	5712	43520	65536	0
10	58963707	1	682	92752	2968064	22347776	33554432

6. k-NEIGHBOR GRAPHS

Definition 6.1. Let $n \equiv 0 \pmod 4$. Let $V_{\mathrm{III}}(n)$ be the set all Type III codes of length n. Let $\Gamma_{\mathrm{III}}^k(n) := (V_{\mathrm{III}}(n), E_{\mathrm{III}}(n))$ be a graph, where any two vertices in $V_{\mathrm{III}}(n)$ are connected by an edge in $E_{\mathrm{III}}(n)$ if and only if they are k-neighbors.

Theorem 6.2. *Let $n \equiv 0 \pmod 4$. Then the graph $\Gamma_{\mathrm{III}}^k(n)$ satisfies the following properties.*

(a) *The number of vertices is $2\prod_{i=1}^{\frac{n}{2}-1}(3^i+1)$.*

(b) *The graph is regular with degree $\dfrac{\prod_{i=0}^{k-1}(3^{n-1-i}-3^{\frac{n}{2}-1})}{\prod_{j=0}^{k-1}(3^{\frac{n}{2}}-3^{\frac{n}{2}-1-j})}$.*

(c) *The number of edges is $\prod_{i=1}^{\frac{n}{2}-1}(3^i+1)\dfrac{\prod_{i=0}^{k-1}(3^{n-1-i}-3^{\frac{n}{2}-1})}{\prod_{j=0}^{k-1}(3^{\frac{n}{2}}-3^{\frac{n}{2}-1-j})}$.*

Proof. Theorems 3.8 and 3.10 shows the statements (a) and (b), respectively. The proof of statement (c) is similar to the proof of Theorem 3.11. □

Definition 6.3. Let $n \equiv 0 \pmod{12}$. Let $V_{\mathrm{III}}(n, \mathbf{1})$ be the set all Type III codes of length n containing $\mathbf{1}$. Let $\Gamma_{\mathrm{III}}^k(n, \mathbf{1}) := (V_{\mathrm{III}}(n, \mathbf{1}), E_{\mathrm{III}}(n, \mathbf{1}))$ be a graph, where any two vertices in $V_{\mathrm{III}}(n, \mathbf{1})$ are connected by an edge in $E_{\mathrm{III}}(n, \mathbf{1})$ if and only if they are k-neighbors.

Theorem 6.4. *Let $n \equiv 0 \pmod{12}$. Then the graph $\Gamma_{\mathrm{III}}^k(n, \mathbf{1})$ satisfies the following properties.*

(a) *The number of vertices is $2\prod_{i=1}^{\frac{n}{2}-2}(3^i+1)$.*

(b) *The graph is regular with degree $\dfrac{\prod_{i=0}^{k-1}(3^{n-2-i}-3^{\frac{n}{2}-1})}{\prod_{j=0}^{k-1}(3^{\frac{n}{2}}-3^{\frac{n}{2}-1-j})}$.*

(c) *The number of edges is $\prod_{i=1}^{\frac{n}{2}-2}(3^i+1)\dfrac{\prod_{i=0}^{k-1}(3^{n-2-i}-3^{\frac{n}{2}-1})}{\prod_{j=0}^{k-1}(3^{\frac{n}{2}}-3^{\frac{n}{2}-1-j})}$.*

Proof. Theorems 4.7 and 4.9 shows the statements (a) and (b), respectively. The proof of statement (c) is similar to the proof of Theorem 4.10. □

Definition 6.5. Let $n \equiv 0 \pmod 2$. Let $V_{\rm IV}(n)$ be the set all Type IV codes of length n. Let $\Gamma^k_{\rm IV}(n) := (V_{\rm IV}(n), E_{\rm IV}(n))$ be a graph, where any two vertices in $V_{\rm IV}(n)$ are connected by an edge in $E_{\rm IV}(n)$ if and only if they are neighbors.

Theorem 6.6. *Let $n \equiv 0 \pmod 2$. Then the graph $\Gamma^k_{\rm IV}(n)$ satisfies the following properties.*

(a) *The number of vertices is $\prod_{i=0}^{\frac{n}{2}-1}(2^{2i+1}+1)$.*
(b) *The graph is regular with degree $\dfrac{\prod_{i=0}^{k-1}(2^{2n-1-2i}-2^{n-1})}{\prod_{j=0}^{k-1}(2^n-2^{n-2-2j})}$.*
(c) *The number of edges is $\frac{1}{2}\prod_{i=0}^{\frac{n}{2}-1}(2^{2i+1}+1)\dfrac{\prod_{i=0}^{k-1}(2^{2n-1-2i}-2^{n-1})}{\prod_{j=0}^{k-1}(2^n-2^{n-2-2j})}$.*

Proof. Theorems 5.8 and 5.10 shows the statements (a) and (b), respectively. The proof of statement (c) is similar to the proof of Theorem 5.11. □

7. An application of neighbors in invariant theory

In this section, we investigate the invariant ring of weight enumerators for Type II codes in genus g, with particular emphasis on identifying the generators of the ring using the concept of neighbors. A binary self-dual code C is called *Type* II if the weight of each codeword of C is a multiple of 4. It is known that a Type II code of length n exists if and only if $n \equiv 0 \pmod 8$. There are 9 Type II codes of length 24 up to equivalence, denoted by C_i for $i = 1, 2, \ldots, 9$. We present these codes in Table 3. For detail discussion about C_i's, we refer the reader to [6, 15, 19].

By d_n and d_n^+, we denote the code with following generator matrices for $n \equiv 0 \pmod 8$:

$$d_n : \begin{pmatrix} 11110000 & \cdots & 0000 \\ 11001100 & \cdots & 0000 \\ 11000011 & \cdots & 0000 \\ \vdots & \ddots & \vdots \\ 11000000 & \cdots & 0011 \end{pmatrix}, \qquad d_n^+ : \begin{pmatrix} 11110000 & \cdots & 0000 \\ 11001100 & \cdots & 0000 \\ \vdots & \ddots & \vdots \\ 11000000 & \cdots & 0011 \\ 10101010 & \cdots & 1010 \end{pmatrix}.$$

In particular, d_8^+ is denoted by e_8. Additionally, g_{24} denotes the binary Golay code of length 24, which is the unique Type II code of this length that does not include any elements of weight 4, see [19].

TABLE 3. Classification of Type II codes of length 24

Code	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
Components	d_{12}^2	$d_{10}e_7^2$	d_8^3	d_6^4	d_{24}	d_4^6	g_{24}	$d_{16}e_8$	e_8^3

Next we recall the definitions and known facts from invariant theory. Here we prefer to denote an element of $\mathbb{F}_2^g$ by a column vector. Let C be a Type II code of length n. Then the *weight enumerator* C in genus g is:

$$W_C^{(g)}(x_a : \ a \in \mathbb{F}_2^g) = \sum_{u,v \in C} \prod_{a \in \mathbb{F}_2^g} x_a^{\,n_a \begin{pmatrix} u_1 \\ \vdots \\ u_g \end{pmatrix}},$$

where $n_a \begin{pmatrix} u_1 \\ \vdots \\ u_g \end{pmatrix}$ is the number of i such that $\begin{pmatrix} u_{1i} \\ \vdots \\ u_{gi} \end{pmatrix} = a$. Now let us use the following notations for various rings in our discussion:

$\mathfrak{B}^{(g)}$: the ring of $W_C^{(g)}$, where C is Type II,
$\mathfrak{D}^{(g)}$: the ring of $W_{d_n^+}^{(g)}$, where $n \equiv 0 \pmod 8$,
$\mathfrak{A}^{(g)}$: the ring of $W_C^{(g)}$, where C is d_n^+ and its neighbors.

Clearly, $\mathfrak{D}^{(g)} \subseteq \mathfrak{A}^{(g)} \subseteq \mathfrak{B}^{(g)}$. Since $\mathfrak{B}^{(g)}$ and $\mathfrak{D}^{(g)}$ are finitely generated over $\mathbb{C}$, see [9, 10], it follows that $\mathfrak{A}^{(g)}$ is as well. It is proved in [10, Proposition 2] that $\mathfrak{D}^{(1)} = \mathfrak{B}^{(1)}$, however $\mathfrak{D}^{(2)}$ is strictly smaller than $\mathfrak{B}^{(2)}$. In this note, we would like to discuss on the ring $\mathfrak{A}^{(g)}$ of weight enumerators of Type II code d_n^+ and its neighbors. Table 4 gives neighbors of code d_n^+ for $n = 8, 16, 24$.

TABLE 4. d_n^+ and its neighbors up to length 24

Code	Neighbors (up to equivalence)
e_8	e_8
d_{16}^+	d_{16}^+
d_{24}^+	C_1, C_5, C_8

Now let us define following matrices in $\mathrm{GL}(2^g,\mathbb{C})$:

$$T_g = \left(\frac{1+i}{2}\right)^g \left((-1)^{(a,b)}\right)_{a,b\in\mathbb{F}_2^g},$$
$$D_S = \mathrm{diag}(i^{S[a]} \text{ for } a \in \mathbb{F}_2^g),$$

where, $S[a] := {}^t aSa$ for any symmetric $g \times g$ matrix S. Let

$$G_g := \langle T_g, D_S, \zeta_8 \rangle$$

be a subgroup of $\mathrm{GL}(2^g,\mathbb{C})$ generated by T_g, D_S and ζ_8, where S runs over all symmetric matrices of order g and $\zeta_8 = e^{2\pi i/8}$ is the primitive 8th root of unity. The order of the group G_g for $g = 1, 2, 3$ are shown in Table 5. The group G_g acts naturally on the polynomial ring $\mathbb{C}[x_a] := \mathbb{C}[x_a : a \in \mathbb{F}_2^g]$. We denote $\mathbb{C}[x_a]^{G_g}$ the invariant ring under the action of G_g.

Table 5. Order of G_g

g	1	2	3
$\lvert G_g \rvert$	192	92160	743178240

We recall [9, 11, 16] for the dimension formulae of the invariant ring $\mathbb{C}[x_a]^{G_g}$ for $g = 1, 2, 3$ as follows:

$$g = 1: \frac{1}{(1-t^8)(1-t^{24})} = 1 + t^8 + t^{16} + 2t^{24} + \cdots,$$
$$g = 2: \frac{1+t^{32}}{(1-t^8)(1-t^{24})^2(1-t^{40})} = 1 + t^8 + t^{16} + 3t^{24} + \cdots,$$
$$g = 3: \frac{\theta(t^8) + t^{352}\theta(t^{-8})}{(1-t^8)(1-t^{16})(1-t^{24})^2(1-t^{40})(1-t^{56})(1-t^{72})(1-t^{120})}$$
$$= 1 + t^8 + 2t^{16} + 5t^{24} + \cdots.$$

where

$$\begin{aligned}\theta(t) := 1 &+ t^3 + 3t^4 + 3t^5 + 6t^6 + 8t^7 + 12t^8 + 18t^9 + 25t^{10} \\ &+ 29t^{11} + 40t^{12} + 50t^{13} + 58t^{14} + 69t^{15} + 80t^{16} \\ &+ 85t^{17} + 96t^{18} + 104t^{19} + 107t^{20} + 109t^{21} + 56t^{22}.\end{aligned}$$

It is known that the invariant ring $\mathbb{C}[x_a]^{G_g}$ is generated by the weight enumerators of Type II codes in genus g, see [9, 11, 21]. In particular, a basis of the vector space generated by the weight enumerators of

Type II code of length 24 in $g = 1, 2, 3$ is given below:

$$g = 1 : W^{(1)}_{C_9}, W^{(1)}_{C_7}$$
$$g = 2 : W^{(2)}_{C_9}, W^{(2)}_{C_7}, W^{(2)}_{C_5}$$
$$g = 3 : W^{(3)}_{C_9}, W^{(3)}_{C_7}, W^{(3)}_{C_5}, W^{(3)}_{C_8}, W^{(3)}_{C_1}.$$

It is can be seen from Table 3 that C_5 is d^+_{24} itself. Moreover, from Table 4 we have C_1, C_5 and C_8 being the neighbors of d^+_{24}. Since $W^{(1)}_{C_9}$ and $W^{(1)}_{C_5}$ are algebraically independent, therefore we have $\mathfrak{A}^{(1)} = \mathbb{C}[W^{(1)}_{e_8}, W^{(1)}_{d^+_{24}}] = \mathbb{C}[x_a]^{G_1}$.

The above discussions and [10, Proposition 2], conclude the following result.

Theorem 7.1. $\mathfrak{D}^{(1)} = \mathfrak{A}^{(1)} = \mathfrak{B}^{(1)}$ *up to the space of degree* 24.

Theorem 7.2. $\mathfrak{D}^{(2)} \subsetneq \mathfrak{A}^{(2)} = \mathfrak{B}^{(2)}$ *up to the space of degree* 24.

Proof. The code d^+_{24} has a neighbor C_1. From the dimension formula, we know that the space of degree 24 in $\mathfrak{B}^{(2)}$ has dimension 3. Since $W^{(2)}_{g_{24}} \notin \mathfrak{A}^{(2)}$, it is enough to show that $W^{(2)}_{C_1}$ belongs to the basis of the space of degree 24 in $\mathfrak{A}^{(2)}$. Therefore, we consider the genus 2 weight enumerators of e_8^3, d^+_{24} and C_1. We select the following monomials of these weight enumerators:

$$\alpha x^{24}_{\substack{0\\0}}, \quad \beta x^{20}_{\substack{0\\0}} x^{4}_{\substack{0\\1}}, \quad \gamma x^{16}_{\substack{0\\0}} x^{8}_{\substack{0\\1}},$$

where α, β and γ represent the coefficients of the monomials. Now we construct the following 3×3 matrix L consisting of the coefficients of the aforementioned 3 monomials from the selected weight enumerators:

Code	α	β	γ
e_8^3	1	42	591
d^+_{24}	1	66	495
C_1	1	30	639

Immediately, $\mathrm{Rank}(L) = 3$. It is known (see [9]) that the ring $\mathfrak{B}^{(2)}$ has the following structure:

$$\mathbb{C}[W^{(2)}_{e_8}, W^{(2)}_{d^+_{24}}, W^{(2)}_{g_{24}}, W^{(2)}_{d^+_{40}}] \oplus \mathbb{C}[W^{(2)}_{e_8}, W^{(2)}_{d^+_{24}}, W^{(2)}_{g_{24}}, W^{(2)}_{d^+_{40}}] W^{(2)}_{d^+_{32}}.$$

Since $\mathrm{Rank}(L) = 3$, the weight enumerators of e_8^3, d^+_{24} and C_1 are algebraically independent. Thus the space of degree 24 in $\mathfrak{A}^{(2)}$ is of dimension 3 and is same as $\mathfrak{B}^{(2)}$. Moreover, $\mathfrak{D}^{(2)} \subsetneq \mathfrak{B}^{(2)}$, see [10]. This completes the proof. □

Our computation shows that the dimensions and a basis of the spaces of degrees 8, 16 and 24 in $\mathfrak{D}^{(3)}$ are as follows:

TABLE 6. Dimension and basis of $\mathfrak{D}^{(3)}$ up to length 24

Length	Dimension	Basis (up to equivalence)
8	1	e_8
16	2	e_8^2, d_{16}^+
24	3	C_9, C_5, C_8

Theorem 7.3. $\mathfrak{D}^{(3)} \subsetneq \mathfrak{A}^{(3)} \subsetneq \mathfrak{B}^{(3)}$ *up to the space of degree* 24.

Proof. From Table 4, we have C_1, C_5 and C_8 are the neighbors of the code d_{24}^+. Clearly, C_5 is d_{24}^+ itself. Since $W_{C_8}^{(3)}$ belongs to the basis of the space of degree 24 in $\mathfrak{B}^{(3)}$, it follows that is true for $\mathfrak{A}^{(3)}$ as well. Since $W_{g_{24}}^{(3)} \notin \mathfrak{A}^{(3)}$, it is enough to show that $W_{C_1}^{(3)}$ belongs to the basis of the space of degree 24 in $\mathfrak{A}^{(3)}$. Therefore, we consider the genus 3 weight enumerators of e_8^3, d_{24}^+, C_8 and C_1. We select the following monomials of these weight enumerators:

$$\alpha x_{\substack{0\\0\\0}}^{20} x_{\substack{0\\1\\1}}^{4}, \quad \beta x_{\substack{0\\1\\0}}^{16} x_{\substack{1\\0\\1}}^{8}, \quad \gamma x_{\substack{0\\0\\1}}^{8} x_{\substack{1\\0\\0}}^{4} x_{\substack{1\\1\\0}}^{12}, \quad \delta x_{\substack{0\\0\\0}}^{4} x_{\substack{0\\1\\0}}^{2} x_{\substack{0\\0\\1}}^{6} x_{\substack{1\\0\\0}}^{6} x_{\substack{1\\1\\1}}^{6},$$

with $\alpha, \beta, \gamma, \delta$ being the coefficients of the monomials. Now we construct the following 4×4 matrix M consisting of the coefficients of the aforementioned 4 monomials from the selected weight enumerators:

Code	α	β	γ	δ
e_8^3	42	591	9491	592704
d_{24}^+	66	495	13860	110800
C_8	42	591	9492	762048
C_1	30	639	7020	659520

It is immediate that $\mathrm{Rank}(M) = 4$. This means $W_{e_8^3}, W_{d_{24}^+}, W_{C_8}, W_{C_1}$ are algebraically independent and form a dimension 4 vector space. Thus the space of degree 24 in $\mathfrak{A}^{(3)}$ is of dimension 4 and hence it is strictly smaller than $\mathfrak{B}^{(3)}$. Moreover, the space of degree 24 in $\mathfrak{D}^{(3)}$ is of dimension 3 (see Table 6), This completes the proof. □

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability statement

The data that support the findings of this study are available from the corresponding author.

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Department of Mathematics, Shahjalal University of Science and Technology, Sylhet-3114, Bangladesh,

Email address: `himadri-mat@sust.edu`

Graduate School of Natural Science and Technology, Kanazawa University, Ishikawa 920-1192, Japan,

Email address: `williamschiari@gmail.com`

Faculty of Science and Engineering, Waseda University, Tokyo 169-8555, Japan,

Email address: `miezaki@waseda.jp`

Faculty of Mathematics and Physics, Kanazawa University, Ishikawa 920-1192, Japan

Email address: `oura@se.kanazawa-u.ac.jp`