Centralizer Algebras of Two Permutation Groups of Order 1344

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Abstract. There are two permutation groups that they share the same character table of order 1344. We take up natural representations on 8 and 14 letters respectively. The purpose of this paper is to examine the semi-simple structure of centralizing algebras in the tensor representation.

1. INTRODUCTION

Let H_1 be a complex reflection group of order 96, which is No. 8 in [6]. It is known that invariant algebra of H_1 is isomorphic to the sub algebra of modular forms for $SL_2(\mathbb{Z})$ using theta functions [2]. Also the invariant algebra of H_1 is a closely related to the algebra of weight enumerators of self-dual and doubly even codes [4]. In [5], we took up this important group H_1 and the centralizer algebras of the tensor representation of H_1 were determined. Additionally, in [3], the multi-matrix structures of the centralizer algebras of the tensor representations of a certain permutation group are discussed.

We continue this line. We take up two permutation groups of order 1344 which have the same character table (*cf.* [8]). Our groups in question are subgroups of the symmetric group of degree 8 and 14 . The purpose of this note is to investigate the centralizer algebras of tensor representations of their permutation representations.

As usual, let $\mathbb{C}$ denote the complex number field. We denote by M_d the matrix algebra of degree d over $\mathbb{C}$. For simplicity, let nM_d denote $\underbrace{M_d \oplus \cdots \oplus M_d}_{n}$.

2. PRELIMINARIES

Schur-Weyl's reciprocity is one the effective methods to find the structure of the centralizer algebra of representation V of an associative algebra. Suppose that a representation (ρ, V) of an associative algebra $\mathcal{A}$ is decomposed into the irreducible ones V_i as follows:

$$V \cong \bigoplus_{i}^{s} m_i V_i$$

Here m_i is the multiplicity of the simple components V_i and s is the number of the essential irreducible representations. The $End_A(V)$, the centralizer algebra

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of A with respect to the representation V is isomorphic to a direct sum of the endomorphism algebras $\mathbb{C}^{m_i}$.

$$End_A(V) \cong \bigoplus_{i}^{s} End_{\mathbb{C}}(\mathbb{C}^{m_i}) \cong \bigoplus_{i}^{s} Mat_{m_i}(\mathbb{C})$$

Thus to find the structure of the centralizer algebra. Let G be a subgroup of symmetric 8, generated by

$$(5,7)(6,8),(2,3,5)(4,7,6),(1,2)(3,4)(5,6)(7,8))(1,5)(2,6)(3,7)(4,8)$$

On the other hand H be a subgroup of symmetric 14, generated by

$$(1,2,3,4,5,6)(14,13,12,11,10,9,8),(1,4,7,9,14,11,8,6)(2,5,13,10)$$

Both G and H are of order 1344. Let $\mathbf{X}$ be the character table and χ_G, χ_H a permutation character. We follow the character table of [8], but switch the 7th and 8th columns. The group G has the following 11 conjugacy classes.

Class	Size	Representative
$\mathfrak{C}_1$	1	1
$\mathfrak{C}_2$	7	$(1,2)(3,4)(5,6)(7,8)$
$\mathfrak{C}_3$	42	$(1,5)(3,7)$
$\mathfrak{C}_4$	42	$(1,3)(2,8)(4,6)(5,7)$
$\mathfrak{C}_5$	84	$(1,5,2)(3,8,7)$
$\mathfrak{C}_6$	168	$(1,2,5,6)(3,4,7,8)$
$\mathfrak{C}_7$	168	$(1,5)(2,4,6,8)$
$\mathfrak{C}_8$	224	$(1,2,7,4)(3,8,5,6)$
$\mathfrak{C}_9$	224	$(1,7)(2,3,6,8,5,4)$
$\mathfrak{C}_{10}$	192	$(2,7,4,8,6,5,3)$
$\mathfrak{C}_{11}$	192	$(2,8,3,4,5,7,6)$

Also the group H has the following 11 conjugacy classes.

Class	Size	Representative
$\mathfrak{C}'_1$	1	1
$\mathfrak{C}'_2$	7	$(1,14)(4,11)(6,9)(7,8)$
$\mathfrak{C}'_3$	42	$(1,7,14,8)(4,6,11,9)$
$\mathfrak{C}'_4$	42	$(1,7,14,8)(2,13)(4,9,11,6)(5,10)$
$\mathfrak{C}'_5$	84	$(1,10,8)(3,6,11)(4,12,9)(5,7,14)$
$\mathfrak{C}'_6$	168	$(1,7)(3,12)(4,6)(5,10)(8,14)(9,11)$
$\mathfrak{C}'_7$	168	$(1,4,7,6,14,11,8,9)(2,5)(3,12)(10,13)$
$\mathfrak{C}'_8$	224	$(1,4,7,9,14,11,8,6)(2,5,13,10)$
$\mathfrak{C}'_9$	224	$(1,5,8,14,10,7)(2,13)(3,6,11)(4,12,9)$
$\mathfrak{C}'_{10}$	192	$(1,2,3,4,5,6,7)(8,14,13,12,11,10,9)$
$\mathfrak{C}'_{11}$	192	$(1,4,7,3,6,2,5)(8,12,9,13,10,14,11)$

Suppose that $\chi_G^{\otimes k}$ and $\chi_H^{\otimes k}$ are decomposed into the irreducible characters as follows:

$$\chi_G^{\otimes k} = d_{G,1}^{(k)}\chi_1 + \cdots + d_{G,11}^{(k)}\chi_{11} \quad \text{for } G$$
$$\chi_H^{\otimes k} = d_{H,1}^{(k)}\chi_1 + \cdots + d_{H,11}^{(k)}\chi_{11} \quad \text{for } H.$$

We would like to find $d_G^{(k)}$ and $d_H^{(k)}$.

3. RESULTS

We follow the argument presented in the paper [8]. It is known that $\chi_G(g)$ (resp. $\chi_H(h)$) is the number of elements which are fixed by $g \in G$ (resp. $h \in H$).

Proposition 3.1. *We have*

$$\chi_G = \chi_1 + \chi_8,$$
$$\chi_H = \chi_1 + \chi_4 + \chi_7.$$

Proof. First we know the following.

	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_6$	$\mathfrak{C}_4$	$\mathfrak{C}_3$	$\mathfrak{C}_5$	$\mathfrak{C}_9$	$\mathfrak{C}_7$	$\mathfrak{C}_8$	$\mathfrak{C}_{10}$	$\mathfrak{C}_{11}$
χ_G	8	0	0	0	4	2	0	2	0	1	1
	$\mathfrak{C}'_1$	$\mathfrak{C}'_2$	$\mathfrak{C}'_6$	$\mathfrak{C}'_3$	$\mathfrak{C}'_4$	$\mathfrak{C}'_5$	$\mathfrak{C}'_9$	$\mathfrak{C}'_7$	$\mathfrak{C}'_8$	$\mathfrak{C}'_{10}$	$\mathfrak{C}'_{11}$
χ_H	14	6	2	6	2	2	0	0	2	0	0

For $d_G^{(1)} = \left(d_{G,1}^{(1)}, \ldots, d_{G,11}^{(1)}\right)$ we have

$$(8,0,0,0,4,2,0,2,0,1,1) = \left(d_{G,1}^{(1)}, \ldots, d_{G,11}^{(1)}\right)\mathbf{X},$$

and for $d_H^{(1)} = \left(d_{H,1}^{(1)}, \ldots, d_{H,11}^{(1)}\right)$ we have

$$(14,6,2,6,2,2,0,0,2,0,0) = \left(d_{H,1}^{(1)}, \ldots, d_{H,11}^{(1)}\right)\mathbf{X}.$$

Then we get

$$d_G^{(1)} = (8,0,0,0,4,2,0,2,0,1,1)\mathbf{X}^{-1}$$
$$= (1,0,0,0,0,0,0,1,0,0,0)$$

and

$$d_H^{(1)} = (14,6,2,6,2,2,0,0,2,0,0)\mathbf{X}^{-1}$$
$$= (1,0,0,1,0,0,1,0,0,0,0).$$

This completes the proof. □

Since χ_G is decomposed into the trivial character and one irreducible character, G is doubly transitive. On the other hand, χ_H is decomposed into the trivial character and two distinct non trivial irreducible characters. Hence H is not doubly transitive, see [7], [1]. Next we calculate $d_G^{(k)}$ and $d_H^{(k)}$.

G	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_6$	$\mathfrak{C}_4$	$\mathfrak{C}_3$	$\mathfrak{C}_5$	$\mathfrak{C}_9$	$\mathfrak{C}_7$	$\mathfrak{C}_8$	$\mathfrak{C}_{10}$	$\mathfrak{C}_{11}$
H	$\mathfrak{C}'_1$	$\mathfrak{C}'_2$	$\mathfrak{C}'_6$	$\mathfrak{C}'_3$	$\mathfrak{C}'_4$	$\mathfrak{C}'_5$	$\mathfrak{C}'_9$	$\mathfrak{C}'_7$	$\mathfrak{C}'_8$	$\mathfrak{C}'_{10}$	$\mathfrak{C}'_{11}$
$\lvert C_G(g)\rvert$	1344	192	16	32	32	6	6	8	8	7	7
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3	3	-1	-1	0	0	0	1	1	$\frac{-1+\sqrt{-7}}{2}$	$\frac{-1-\sqrt{-7}}{2}$
χ_3	3	3	-1	-1	0	0	0	1	1	$\frac{-1-\sqrt{-7}}{2}$	$\frac{-1+\sqrt{-7}}{2}$
χ_4	6	6	2	2	2	0	0	0	0	-1	-1
χ_5	7	7	-1	-1	-1	1	1	-1	-1	0	0
χ_6	8	8	0	0	0	-1	-1	0	0	1	1
χ_7	7	-1	-1	3	-1	1	-1	-1	1	0	0
χ_8	7	-1	-1	-1	3	1	-1	1	-1	0	0
χ_9	14	-2	-2	2	2	-1	1	0	0	0	0
χ_{10}	21	-3	1	1	-3	0	0	1	-1	0	0
χ_{11}	21	-3	1	-3	1	0	0	-1	1	0	0

Consider the following matrix A_G such that

$$\begin{pmatrix} \chi_1\chi \\ \chi_2\chi \\ \vdots \\ \chi_{11}\chi \end{pmatrix} = A_G \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_{11} \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} \chi_1(\mathfrak{C}_i)\chi(\mathfrak{C}_i) \\ \chi_2(\mathfrak{C}_i)\chi(\mathfrak{C}_i) \\ \vdots \\ \chi_{11}(\mathfrak{C}_i)\chi(\mathfrak{C}_i) \end{pmatrix} = A_G \begin{pmatrix} \chi_1(\mathfrak{C}_i) \\ \chi_2(\mathfrak{C}_i) \\ \vdots \\ \chi_{11}(\mathfrak{C}_i) \end{pmatrix},$$

$$\begin{pmatrix} \chi_1(\mathfrak{C}_1) & \chi_1(\mathfrak{C}_2) & \chi_1(\mathfrak{C}_6) & \dots & \chi_1(\mathfrak{C}_{11}) \\ \chi_2(\mathfrak{C}_1) & \chi_2(\mathfrak{C}_2) & \chi_2(\mathfrak{C}_6) & \dots & \chi_2(\mathfrak{C}_{11}) \\ \chi_3(\mathfrak{C}_1) & \chi_3(\mathfrak{C}_2) & \chi_3(\mathfrak{C}_6) & \dots & \chi_3(\mathfrak{C}_{11}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \chi_{11}(\mathfrak{C}_1) & \chi_{11}(\mathfrak{C}_2) & \chi_{11}(\mathfrak{C}_6) & \dots & \chi_{11}(\mathfrak{C}_{11}) \end{pmatrix} \begin{pmatrix} \chi(\mathfrak{C}_1 & & & & & \\ & \chi(\mathfrak{C}_2) & & & & \\ & & \chi(\mathfrak{C}_6) & & & \\ & & & \chi(\mathfrak{C}_4) & & \\ & & & & \ddots & \\ & & & & & \chi(\mathfrak{C}_{11}) \end{pmatrix}$$

$$= A_G \begin{pmatrix} \chi_1(\mathfrak{C}_1)\dots\chi_1(\mathfrak{C}_{11}) \\ \vdots \\ \chi_{11}(\mathfrak{C}_1)\dots\chi_{11}(\mathfrak{C}_{11}) \end{pmatrix}.$$

And for A_H we get

$$\begin{pmatrix} \chi_1(\mathfrak{C}'_1) & \chi_1(\mathfrak{C}'_2) & \chi_1(\mathfrak{C}'_6) & \dots & \chi_1(\mathfrak{C}'_{11}) \\ \chi_2(\mathfrak{C}'_1) & \chi_2(\mathfrak{C}'_2) & \chi_2(\mathfrak{C}'_6) & \dots & \chi_2(\mathfrak{C}'_{11}) \\ \chi_3(\mathfrak{C}'_1) & \chi_3(\mathfrak{C}'_2) & \chi_3(\mathfrak{C}'_6) & \dots & \chi_3(\mathfrak{C}'_{11}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \chi_{11}(\mathfrak{C}'_1) & \chi_{11}(\mathfrak{C}'{}_2) & \chi_{11}(\mathfrak{C}'_6) & \dots & \chi_{11}(\mathfrak{C}'_{11}) \end{pmatrix} \begin{pmatrix} \chi(\mathfrak{C}'_1) & & & & & \\ & \chi(\mathfrak{C}'_2) & & & & \\ & & \chi(\mathfrak{C}'_6) & & & \\ & & & \chi(\mathfrak{C}'_3) & & \\ & & & & \ddots & \\ & & & & & \chi(\mathfrak{C}'_{11}) \end{pmatrix}$$
$$= A_H \begin{pmatrix} \chi_1(\mathfrak{C}'_1) \dots \chi_1(\mathfrak{C}'_{11}) \\ \vdots \\ \chi_{11}(\mathfrak{C}'_1) \dots \chi_{11}(\mathfrak{C}'_{11}) \end{pmatrix}.$$

Thus we have

$$\begin{aligned} \mathbf{X} diag(8,0,0,0,4,2,0,2,0,1,1) &= A_G \mathbf{X} \\ A_G &= \mathbf{X} diag(8,0,0,0,4,2,0,2,0,1,1)\mathbf{X}^{-1}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{X} diag(14,6,2,6,2,2,0,0,2,0,0) &= A_H \mathbf{X} \\ A_H &= \mathbf{X} diag(14,6,2,6,2,2,0,0,2,0,0)\mathbf{X}^{-1}. \end{aligned}$$

Therefore we have

$$\begin{aligned} d_G^{(k)} &= d_G^{k-1} A_G \\ &= d_G^{(1)} A_G^{k-1} \\ &= d_G^{(1)} X \left(diag(8,0,0,0,4,2,0,0,2,1,1)\right)^{k-1} X^{-1} \end{aligned}$$

Subsequently, we possess

$$
\begin{aligned}
a_{G,k} &= \frac{2^{3k}}{1344} + \frac{2^{2k}}{32} + \frac{7}{24}2^k + \frac{2}{7},\\
b_{G,k} &= \frac{2^{3k}}{448} - \frac{2^{2k}}{32} + \frac{2^k}{8} - \frac{1}{7},\\
c_{G,k} &= \frac{2^{3k}}{448} - \frac{2^{2k}}{32} + \frac{2^k}{8} - \frac{1}{7},\\
d_{G,k} &= \frac{2^{3k}}{224} + \frac{2^{2k}}{16} - \frac{2}{7},\\
e_{G,k} &= \frac{2^{3k}}{192} - \frac{2^{2k}}{32} + \frac{2^k}{24},\\
f_{G,k} &= \frac{2^{3k}}{168} - \frac{2^k}{6} + \frac{2}{7},\\
g_{G,k} &= \frac{2^{3k}}{192} - \frac{2^{2k}}{32} + \frac{2^k}{24},\\
h_{G,k} &= \frac{2^{3k}}{192} + \frac{3}{32}2^{2k} + \frac{7}{24}2^k,\\
i_{G,k} &= \frac{2^{3k}}{96} + \frac{2^{2k}}{16} - \frac{2^k}{6},\\
j_{G,k} &= \frac{2^{3k}}{64} - \frac{3}{32}2^{2k} + \frac{2^k}{8},\\
l_{G,k} &= \frac{2^{3k}}{64} + \frac{2^{2k}}{32} - \frac{2^k}{8}.
\end{aligned}
$$

and for χ_H

$$
\begin{aligned}
d_H^{(k)} &= d_H^{k-1} A_H\\
&= d_H^{(1)} A_H^{k-1}\\
&= d_H^{(1)} X \left(diag(14,6,2,6,2,2,0,2,0,0,0)\right)^{k-1} X^{-1}
\end{aligned}
$$

Afterwards

$$a_{H,k} = 2^k\left(\frac{1}{1344}\cdot 7^k + \frac{7}{192}\cdot 3^k + \frac{37}{96}\right),$$
$$b_{H,k} = 2^k\left(\frac{1}{448}\cdot 7^k - \frac{1}{64}\cdot 3^k + \frac{1}{32}\right),$$
$$c_{H,k} = 2^k\left(\frac{1}{448}\cdot 7^k - \frac{1}{64}\cdot 3^k + \frac{1}{32}\right),$$
$$d_{H,k} = 2^k\left(\frac{1}{224}\cdot 7^k + \frac{3}{32}\cdot 3^k + \frac{3}{16}\right),$$
$$e_{H,k} = 2^k\left(\frac{1}{192}\cdot 7^k + \frac{1}{192}\cdot 3^k - \frac{5}{96}\right),$$
$$f_{H,k} = 2^k\left(\frac{1}{168}\cdot 7^k + \frac{1}{24}\cdot 3^k - \frac{1}{6}\right),$$
$$g_{H,k} = 2^k\left(\frac{1}{192}\cdot 7^k + \frac{17}{192}\cdot 3^k + \frac{19}{96}\right),$$
$$h_{H,k} = 2^k\left(\frac{1}{192}\cdot 7^k - \frac{7}{192}\cdot 3^k + \frac{7}{96}\right),$$
$$i_{H,k} = 2^k\left(\frac{1}{96}\cdot 7^k + \frac{5}{96}\cdot 3^k - \frac{11}{48}\right),$$
$$j_{H,k} = 2^k\left(\frac{1}{64}\cdot 7^k + \frac{1}{64}\cdot 3^k - \frac{5}{32}\right),$$
$$l_{H,k} = 2^k\left(\frac{1}{64}\cdot 7^k - \frac{7}{64}\cdot 3^k + \frac{7}{32}\right).$$

Theorem 3.2. *We have*

$$\mathfrak{A}_G^{(k)} \cong \begin{cases} 2M_1 \\ M_{a_{G,k}} \oplus 2M_{b_{G,k}} \oplus M_{d_{G,k}} \oplus 2M_{e_{G,k}} \oplus M_{f_{G,k}} \oplus M_{h_{G,k}} \oplus M_{i_{G,k}} \oplus M_{j_{G,k}} \oplus M_{l_{G,k}} \end{cases}$$

and $\mathfrak{A}_H^{(k)} \cong \begin{cases} 3M_1 \\ M_{a_{H,k}} \oplus 2M_{b_{H,k}} \oplus M_{d_{H,k}} \oplus M_{e_{H,k}} \oplus M_{f_{H,k}} \oplus M_{g_{H,k}} \oplus M_{h_{H,k}} \oplus M_{i_{H,k}} \oplus M_{j_{H,k}} \oplus M_{l_{H,k}} \end{cases}$

where

$$a_{G,k}, b_{G,k}, c_{G,k}, d_{G,k}, e_{G,k}, f_{G,k}, g_{G,k}, h_{G,k}, i_{G,k}, j_{G,k}, l_{G,k}$$

and

$$a_{H,k}, b_{H,k}, c_{H,k}, d_{H,k}, e_{H,k}, f_{H,k}, g_{H,k}, h_{H,k}, i_{H,k}, j_{H,k}, l_{H,k}$$

are given above.

Corollary 3.3. *We have*

$$\dim \mathfrak{A}_G^{(k)} = \frac{1}{1344}\cdot 2^{6k} + \frac{1}{32}\cdot 2^{4k} + \frac{7}{24}\cdot 2^{2k} + \frac{2}{7}$$

and

$$\dim \mathfrak{A}_H^{(k)} = 2^{2k}\left(\frac{1}{1344}\cdot 7^{2k} + \frac{7}{192}\cdot 3^{2k} + \frac{37}{96}\right).$$

The second assertion of Corollary 3.3 is obtained by taking the square sum of the dimensions of the simple components. We conclude this paper with a small table of dim $\mathfrak{A}$.

	1	2	3	4	5	6
$\dim \mathfrak{A}_G^{(k)}$	2	16	342	14606	831982	51656046
$\dim \mathfrak{A}_H^{(k)}$	3	82	7328	1159392	217424128	42262333952

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REFERENCES

[1] Bannai, E., Ito, T., Algebraic Combinatorics I: Association Schemes, Benjamin (1984).
[2] Broué, M., Enguehard, M., Polynomes des poids de certains codes et fonctions thota de certains reseaux, Ann. Sci. Ecole Norm. Sup. (4) **5** (1972), 157-181.
[3] Imamura, H., Kosuda, M., Oura, M., Note on the permutation group associated to E-polynomials, J. Algebra Comb. Discrete Appl (2021), no 9, 1-7.
[4] Gleason, A.M., Weight polynomials of self-dual codes and the MacWilliams identities, Actes du Congres International des Mathematiciens (Nice, 1970), Tome 3, pp. 211-215. Gauthier-Villars. Paris, 1971.
[5] Kosuda, M., Oura, M., On the centralizer algebras of the primitive unitary reflection group of order 96, Tokyo J. Math. 39 (2016), no. 2, 469-482.
[6] Shephard, G.C., Todd, J. A., Finite unitary reflection groups, Canadian J. Math., **6**, (1954), 274-304.
[7] Wielandt, H., Finite permutation groups, Academic Press, New York-London 1964.
[8] Yoshiara, S, An Example of Non-isomorphic Group Association Schemes with the same Parameters, Europ. J. Combinatorics (1997), no 18, 721-738.

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